MinMaxGDJS : A web toolbox to handle pseudo-periodic series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ semiring

Renato Markele Ferreira Cândido *,** **Mehdi Lhommeau** ** **Laurent Hardouin** ** **Rafael Santos Mendes** *

* *Laboratoire Angevin de Recherche en Ingénierie des Systèmes, Université d'Angers, France (e-mails: mehdi.lhommeau@univ-angers.fr, laurent.hardouin@univ-angers.fr)*
** *Department of Computer Engineering and Industrial Automation - DCA, School of Electrical and Computer Engineering, State University of Campinas - UNICAMP - 13083-970 Campinas, SP, Brazil (e-mails: renatomarkele@hotmail.com, rafael@dca.fee.unicamp.br)*

Abstract: Timed Event Graphs (TEGs) constitue an important class of Discrete Event Systems that have a wide domain of applicability. Analysing the temporal behavior of these systems has proven to be efficient, primarily through the use of max-plus algebra and more particularly with formal series of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ semiring. We present, MinMaxGDJS, a web toolbox to handle formal series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ semiring. Making the library accessible on the web makes it easier to experiment and calculus with formal series without the overhead of configuring and building the C++ library. Web browsers become an excellent platform for giving portable demos.

Keywords: $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ semiring, (max,+) algebra, Timed Event Graphs, formal series

1. INTRODUCTION

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Since the beginning of the 80s, it has been know that the class of Timed Event Graphs (TEGs) can be studied thanks to linear models in some specific algebraic structures called semirings. Semiring $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, introduced by the (max,+)-team of INRIA Rocquencourt Baccelli et al. (1992), is one of the algebraic structures in which we can manipulate TEG transfer as formal series in two variables γ and δ. This paper presents the web software toolbox MinMaxGDJS to hande formal series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ semiring. The paper is organized as follows. In section 2 Timed Event Graphs is first presented. Then, the modeling in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is presented. Section 3 is devoted to the presentation of computer tools for the calculation of formal series. Eventually, the question of control synthesis is addressed in section 4 after some reminders on the residuation theory and its application to $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ semiring. All the scripts used to do the calculations are given in the text.

2. TIMED EVENT GRAPHS IN $\mathcal{M}_{IN}^{AX}[\![\gamma,\delta]\!]$

Let us recall that a Timed Event Graph (TEG) is a Petri net whose each place has only one upstream transition and only one downstream transition. Consider the TEG depicted in the Fig.2 as an example. It describes a model for a workshop with 3 machines (M_1, M_2 and M_3). Machines M_1 and M_2 can process parts which are assembled by machine M_3. Inputs u_1 and u_2 represent the admission dates of parts into the system and the transportation time are t_{u_1} and t_{u_2} for the respective inputs; machine M_1 can process two parts in t_1 time units; machine M_2 can process one part in t_2 time units. The transportation time between machines M_1 and M_3 is t_{13} time units and between machines M_2 and the one M_3 is t_{23} time units. Machine M_3 can assemble one part in t_3 time units. The behaviour of these graphs can be described by using linear equations system into the semiring $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Hereafter, the semiring $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ (see Baccelli et al. (1992) for more details) and TEG description in this semiring are recalled.

2.1 Semiring $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

Semiring $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is formally the quotient semiring $\mathbb{B}[\![\gamma,\delta]\!]$ (the set of formal power series in two commutative variable γ and δ, with Boolean coefficients and with exponents in $\mathbb{Z}$), by the equivalence relation $x\mathcal{R}y \iff \gamma^*(\delta^{-1})^*x = \gamma^*(\delta^{-1})^*y$, where $a^* = \bigoplus_{i\in\mathbb{N}} a^i$ (Kleen star operator) with $a^0 = e = \gamma^0\delta^0$ the neutral element for the product. Semiring $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is complete, this means that it is closed under infinite sums and multiplication distributes over infinite sums.

As $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is a quotient semiring, an element of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ may admit several representatives in $\mathbb{B}[\![\gamma,\delta]\!]$. The representative which is minimal with respect to the number of terms is called the minimum representative.

A simple geometrical interpretation of the previous equivalence relation is available in the (γ,δ)-place. Consider a monomial $\gamma^k\delta^t \in \mathbb{B}[\![\gamma,\delta]\!]$, its south-east cone is defined as $\{(k',t') \mid k' \geq k \text{ and } t' \leq t\}$. The south-east cone of a series in $\mathbb{B}[\![\gamma,\delta]\!]$ is defined as the union of the south-east cones associated with the monomials composing the considered series. For two elements s_1 and s_2 in $\mathbb{B}[\![\gamma,\delta]\!]$, $s_1\mathcal{R}s_2$ (*i.e.*, s_1 and s_2 are equal in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$) is equivalent to the equality of their south-east cones. Direct consequences of the previous geometrical interpretation are :

- simplification rules in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$
$$\gamma^k \oplus \gamma^l = \gamma^{\min(k,l)} \text{ and } \delta^k \oplus \delta^l = \delta^{\max(k,l)}. \quad (1)$$
- a simple formulation of the order relation for monomials
$$\gamma^n\delta^t \preceq \gamma^{n'}\delta^{t'} \iff n \geq n' \text{ and } t \leq t'. \quad (2)$$

The dater canonically associated with the series s in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is the unique non-decreasing function $d_s : \mathbb{Z} \to \mathbb{Z} \cup \{-\infty, +\infty\}$ such that $s = \bigoplus_{k\in\mathbb{Z}} \gamma^k\delta^{d_s(k)}$. A simple interpretation of the variables γ and δ for daters is available :

- multiplying a series s by γ is equivalent to shifting the argument of the associated dater function by -1 ;
- multiplying a series s by δ is equivalent to shifting the values of the associated date function by 1.

Example 1. Consider the series $s = \gamma\delta \oplus \gamma^3\delta^4 \oplus \gamma^4\delta^3$ represented by dots in Fig. 1. The south-east cone of s is colored in grey. The minimum representative of s in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is $s = \gamma\delta \oplus \gamma^3\delta^4$. This result could be obtained using the simplification rules of (1).

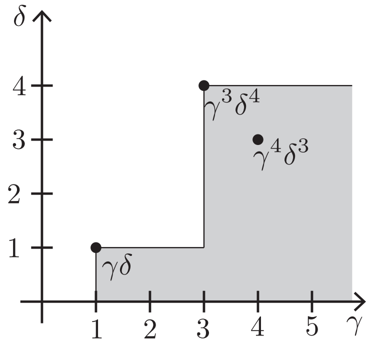

Fig. 1. Series s and its south-east cone (grey)

Besides,
$$s = \bigoplus_{k\leq 0} \gamma^k\delta^{-\infty} \oplus \bigoplus_{k=1,2} \gamma^k\delta \oplus \bigoplus_{k\geq 3} \gamma^k\delta^4. \quad (3)$$
Therefore, the dater d_s associated with s is given by
$$d_s(k) = \begin{cases} -\infty & \text{if } k \leq 0 \\ 1 & \text{if } k = 1,2 \\ 4 & \text{if } k \geq 3 \end{cases}$$

2.2 Linear state-space representation of TEG in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

The dynamics of a TEG may be captured by associating each transition with a series $s \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, where $d_s(k)$ is defined as the time of firing k of the transition. Therefore, for TEG, γ is a shift operator in the event domain, where an event is interpreted as the firing of the transition, and δ is a shift operator in the time domain. The transitions of a TEG are divided into three categories :

- state transitions $(x_1, \ldots, x_n)$: transitions with at least one input place and one output place ;
- input transitions $(u_1, \ldots, u_p)$: transitions with at least one output, but no input places ;
- output transitions $(y_1, \ldots, y_m)$: transitions with at least one input place, but no output places.

Under the earliest functioning rule (*i.e.*, state and output transitions fire as soon as they are enabled), with respect to a place with initially m tokens and holding time t, the influence of its upstream transition on its downstream transition is a positive shift in the time domain of t time units and a negative shift in the event domain of m events. The complete shift operator is coded by the monomial $\gamma^m\delta^t$ in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Therefore, consider the place upstream from transition t_i and downstream from transition t_j, the influence of transition t_j on transition t_i is coded by the monomial f_{ij} in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ defined by $f_{ij} = \gamma^{m_{ij}}\delta^{\tau_{ij}}$ where m_{ij} is the initial number of tokens in the place and τ_{ij} is the holding time of the place.

Consequently, a TEG admits a linear state-space representation in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$:
$$\begin{cases} X = AX \oplus BU \\ Y = CX \end{cases} \quad (4)$$
where $X \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^n$ is the state, $U \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^p$ the input and $Y \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^m$ the output. $A \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times n}$, $B \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times p}$ and $C \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{m\times n}$ are matrices with monomial entries describing the influence of transitions on each other.

We recall that the smallest solution of equation $X = AX \oplus BU$ is $X = A^*BU = (B \oplus AB \oplus A^2B \oplus \ldots)U$. Under the earliest functioning rule, it is possible to express an input-output relation, H, given by $H = CA^*B$, then
$$Y = CA^*BU = HU. \quad (5)$$
Each entries of matrix H is a pseudo-periodic series admitting a representation of the form : $s = p \oplus qr^*$ with $p = \bigoplus_{i=0}^{\alpha} \gamma^{n_i}\delta^{t_i}$, $q = \bigoplus_{j=0}^{\beta} \gamma^{N_j}\delta^{T_j}$ and $r = \gamma^\nu\delta^\tau$. The series is said proper if $(n_\alpha, t_\alpha) \leq (N_0, T_0)$ and if $(N_\beta - N_0, T_\beta - T_0) \leq (\nu, \tau)$. A series admits a simplest periodic proper representation, called the canonical form of s.

3. TOOLS

The team (max, +) of the INRIA Rocquencourt has developed a toolbox, integrated in software Scilab, for calculation in the semirings (max, +) or (min, +). This toolbox implements various algorithms (sum, product, Kleene star, residuation, eigenvalues and eigenvectors) for max-plus matrices. We propose here to present a WEB library, MinMaxGDJS, of calculation for the formal series in the semiring $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. This library is based on MinMaxGD (see Cottenceau et al. (2000)). MinMaxGD is a C++ library to handle power series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. In general, these power series become too difficult to handle by hand. But they can easily be handled with MinMaxGD library. The algorithms proposed in MinMaxGD toolbox are initiated in 1992 by Gaubert (1992). It is still in evolution in order to be improved until today. This library is freely available Hardouin et al. (2015) but requires to have development skills in C++.

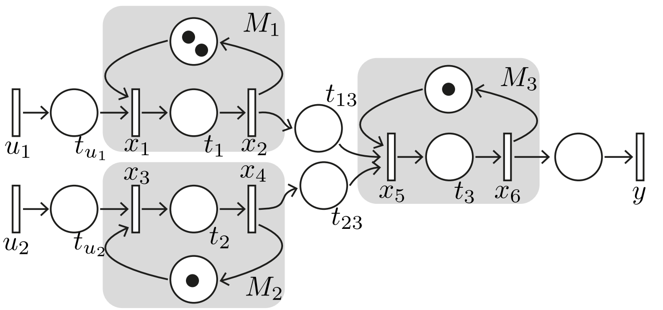

Fig. 2. Workshop system

3.1 MinMaxGDJS

Wee see today, that JavaScript is used everywhere from the browser to the server, including desktops and mobile devices. Web browsers have become an increasingly attractive platform for application developers. Browsers make it comparatively easy to deliver cross-platform applications, because they are effectively ubiquitous. Practically all computing platforms from desktops and tablets to mobile phones-ship with web browsers. But rewriting the MinMaxGD code into Javascript for web usage is a very long and difficult process. For this reason, we have used Emscripten (see Zakai (2014)), a special compiler that "compiles" C/C++ code to highly optimizable Javascript, provides an "easy" way to port the MinMaxGD library to the web. An experimental porting of MinMaxGD library is performed in our work. Our goal is to enable the largest possible number of users to handle the formal series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ semiring without having to make tedious manipulation. This experimental porting is available `http://perso-laris.univ-angers.fr/~lhommeau/minmaxgdcalc.html`.

4. APPLICATION

The objective of this section is to illustrate the series manipulation in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ semiring with MinMaxGDJS in the context of a control problem for TEG.

4.1 Model description in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

To this end, in the TEG depicted in Figure 2, the times are chosen as: $t_1 = 5, t_2 = 5, t_3 = 20, t_{13} = t_{23} = 2, t_{u_1} = 6$ and $t_{u_2} = 12$. One must observe that in this situation where machine M_1 has the greatest production rate, which is equal to $\frac{2}{5}$ and M_3 has the smallest, which is equal to $\frac{1}{20}$. We can also observe that the system can be unstable since the number of tokens in the places among M_1 and M_2 and M_3 can be unbounded. This is the case for the system impulse response, that is, when all the supply materials are available at date $t = 0$.

Then, this TEG can be modelled in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ with

$$\begin{cases} x_1 = \gamma^2 x_2 \oplus \delta^6 u_1 \\ x_2 = \delta^5 x_1 \\ x_3 = \gamma x_4 \oplus \delta^{12} u_2 \\ x_4 = \delta^5 x_3 \\ x_5 = \delta^2 x_2 \oplus \delta^2 x_4 \oplus \gamma x_6 \\ x_6 = \delta^{20} x_5 \\ y = x_6 \end{cases}$$

and its state space representation is given by

$$\begin{cases} X = \begin{pmatrix} \varepsilon & \gamma^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \delta^5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \delta^5 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \delta^2 & \varepsilon & \delta^2 & \varepsilon & \gamma \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^{20} & \varepsilon \end{pmatrix} X \oplus \begin{pmatrix} \delta^6 & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \delta^{12} \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} U \ , \\ Y = (\varepsilon\ \varepsilon\ \varepsilon\ \varepsilon\ \varepsilon\ e)\, X \end{cases}$$

where $\varepsilon = \gamma^{+\infty}\delta^{-\infty}$ (resp. $e = \gamma^0\delta^0$) is the zero (resp. unit) element of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ semiring.

Using MinMaxGDJS (see Figure 3) we can compute, using (5), the transfer matrix H :

$$H = \left(\delta^{33}(\gamma\delta^{20})\ \delta^{39}(\gamma\delta^{20})\right).$$

4.2 Optimal Open-Loop Control

Formally, the optimal control problem consists in computing the greatest control U such that

$$Y = CA^*BU = HU \preceq Z. \tag{6}$$

Since, the left product is residuated [1] the following equivalence holds

$$Y = CA^*BU = HU \iff U \preceq H \backslash\!\!\circ Z. \tag{7}$$

In other words, the greatest control achieving the objective is

$$U_{opt} = H \backslash\!\!\circ Z. \tag{8}$$

The reference trajectory Z is assumed to be known

$$Z = \delta^{80} \oplus \gamma\delta^{90} \oplus \gamma^3\delta^{100} \oplus \gamma^4\delta^{+\infty}.$$

It models that 1 part is desired before or at time 80 (recall that the first token is labelled 0), 2 parts before (or at) time 90 and 4 parts before (or at) time 100. It must be noted that the numbering of parts starts at 0. According to Equation (8), the optimal firing trajectories of the two inputs of the TEG are as follows:

$$U_{opt} = H \backslash\!\!\circ Z = \begin{pmatrix} \delta^7 \oplus \gamma\delta^{27} \oplus \gamma^2\delta^{47} \oplus \gamma^3\delta^{67} \oplus \gamma^4\delta^{\infty} \\ \delta \oplus \gamma\delta^{21} \oplus \gamma^2\delta^{41} \oplus \gamma^3\delta^{61} \oplus \gamma^4\delta^{\infty} \end{pmatrix}$$

and the resulting optimal output, $Y_{opt} = HU_{opt}$, is

$$Y_{opt} = \delta^{40} \oplus \gamma\delta^{60} \oplus \gamma^2\delta^{80} \oplus \gamma^3\delta^{100} \oplus \gamma^4\delta^{\infty},$$

that means that the first part will exit the system at time 40, the second one at time 60, the third one at time 80 and the fourth one at time 100. This output trajectory is the greatest in the image of matrix CA^*B, that is the greatest reachable output, such that the events occur before the desired dates. Figure (4) shows the two trajectories of the formal series Z and Y_{opt}. This example was computed with library MinMaxgdGDJS (see Figure 5).

[1] *i.e.* for all $Z \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^m$ the subset $\{U \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^p \mid HU \preceq Z\}$ admits a maximal element

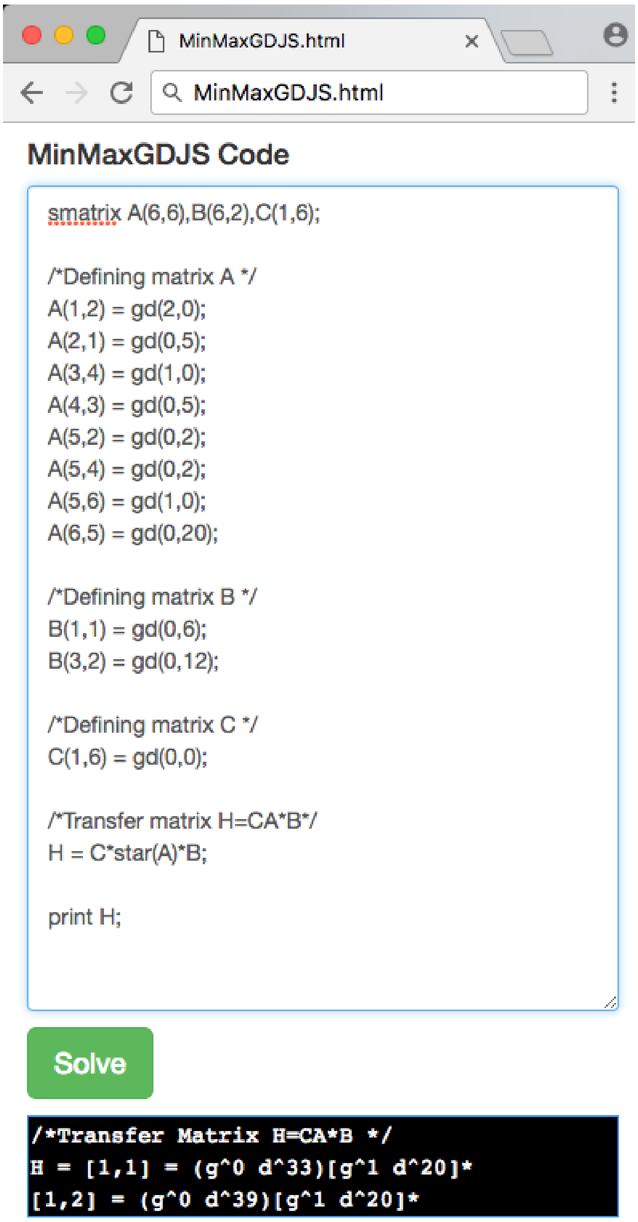

Fig. 3. MinMaxGDJS script to compute matrix H

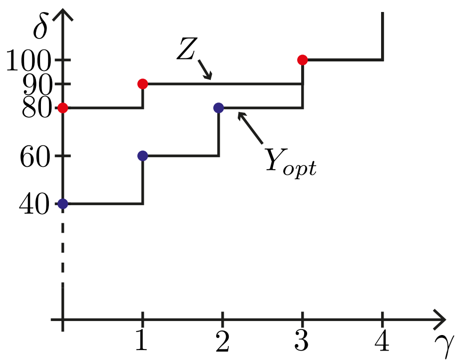

Fig. 4. Z and U_{opt} trajectories

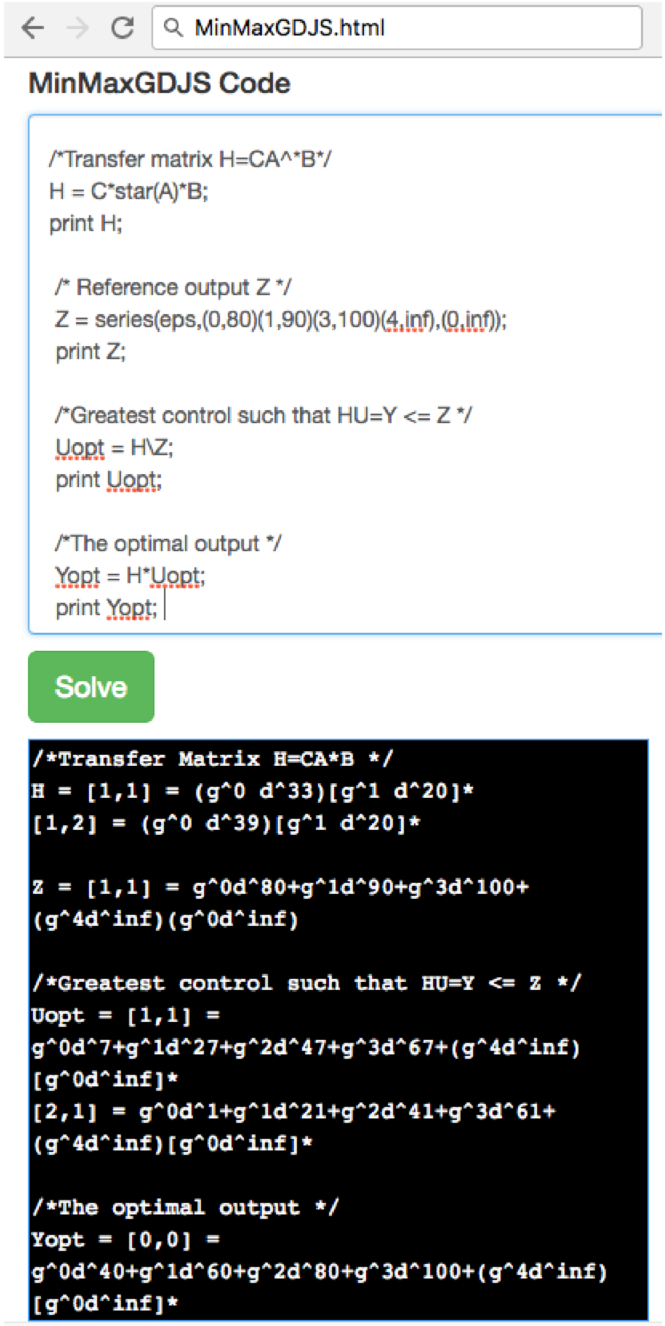

Fig. 5. MinMaxGDJS script to compute matrix U_{opt}

Over the past three decades, many fundamental problems for max-plus linear systems have been studied by researchers, for example, controllability (Prou and Wagneur (1999)), observability (Hardouin et al. (2010)), and the model reference control problem (Cottenceau et al. (2001)).

5. CONCLUSION

We have presented MinMaxGDJS, a WEB library, based on the C++ Library MinMaxGD, to handle formal series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ semiring. We have shown, through an example of just-in-time TEG control, how to use some features of this library. MinMaxGDJS encourages the experiments and especially calculations with pseudo-periodic series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

REFERENCES

Baccelli, F., Cohen, G., Olsder, G., and Quadrat, J. (1992). *Synchronization and Linearity : An Algebra for Discrete Event Systems*. Wiley and Sons.

Cottenceau, B., Hardouin, L., Boimond, J.L., and Ferrier, J.L. (2001). Model reference control for timed event graphs in dioids. *Automatica*, 37(9), 1451–1458. doi: 10.1016/S0005-1098(01)00073-5.

Cottenceau, B., Lhommeau, M., Hardouin, L., and Boimond, J.L. (2000). Data processing tool for calculation in dioid. In *WODES'2000, Workshop on Discrete Event Systems*, xx.

Gaubert, S. (1992). *Théorie des Systèmes Linéaires dans les Dioïdes*. Thèse, École des Mines de Paris.

Hardouin, L., Cottenceau, B., and Lhommeau, M. (2015). MinMaxGD library. URL `http://perso-laris.univ-angers.fr/~hardouin/outils.html`. [accessed 3-November-2016].

Hardouin, L., Maia, C.A., Cottenceau, B., and Lhommeau, M. (2010). Observer design for (max,+) linear systems. *IEEE Transactions on Automatic Control*, 55(2), 538 – 543.

Prou, J.M. and Wagneur, E. (1999). Controllability in the max-algebra. *Kybernetika*, 35, 13–24.

Zakai, A. (2014). *Emscripten: An LLVM-to-JavaScript Compiler*. `https://github.com/kripken/emscripten`.