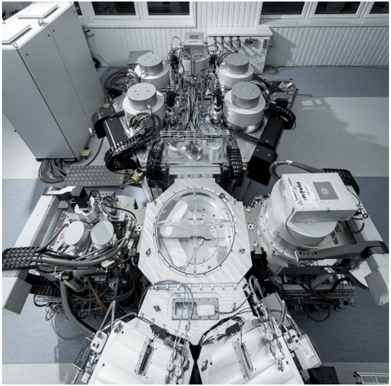
# The Tropical Regulation Problem

V. M. Gonçalves



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- For instance, in the manufacture of semiconductors - pictured in the first slide - it can be desirable that a wafer must be unloaded from the first module in at most 10 time units after the process in this module is finished . Otherwise, residual heat inside may damage the wafer. However, it cannot be unloaded till the next module is free... See [6];

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- Synchronization problem.

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- $Im\{M\}$ is the *right image* of $M$, that is, the set $\{x | \exists y \ , \ \ x = My\}$;

# Problem statement

- Consider a *Tropical Linear Event-Invariant System*

$$x[k+1] = Ax[k] \oplus Bu[k] \tag{1}$$

in which $x[k] \in \mathcal{X}$ is the *state vectors* and $u[k] \in \mathcal{U}$ is the *control vectors*;

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- $\mathcal{S}$ is the set of *desirable specifications*;
- **Tropical regulation problem** $\mathcal{R}(A, B, E, D)$: find a control action $u[k]$ such that for *every* initial condition $x[0]$ there exists a natural number $K$ such that $x[k] \in \mathcal{S}$ for all $k \geq K$;

# (A,B) geometrical invariance and controllable coupled problems

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- Given a specification semimodule $\mathcal{S}$ of a problem $\mathcal{R}(A, B, E, D)$, there exists a maximal $(A, B)$ geometrical invariant set inside $\mathcal{S}$. It will be denoted by $\mathcal{K}_{max}(\mathcal{R})$;

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- May seems restrictive, but it is not.

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- Consider the problem $\mathcal{R}(A, B, E, D)$

$$\begin{pmatrix} x_1[k+1] \\ x_2[k+1] \\ x_3[k+1] \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \perp & \perp \\ \perp & 0 & \perp \\ \perp & \perp & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{pmatrix} \oplus \underbrace{\begin{pmatrix} 0 & \perp & \perp \\ \perp & 0 & \perp \\ \perp & \perp & 0 \end{pmatrix}}_{B} \begin{pmatrix} u_1[k] \\ u_2[k] \\ u_3[k] \end{pmatrix} \tag{3}$$

in which the specification is

$$\underbrace{\begin{pmatrix} 0 & \perp & \perp \end{pmatrix}}_{E} \begin{pmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{pmatrix} = \underbrace{\begin{pmatrix} \perp & 0 & \perp \end{pmatrix}}_{D} \begin{pmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{pmatrix}. \tag{4}$$

*i.e: three completely independent machines* in which it is desirable to synchronize the first with the second, but we dont care about the third one;

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- Do this problem makes sense as a whole?
- Suggestion: consider the subproblem with the two machines + constraints, disregard the last machine. This subproblem *is coupled*;

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- There are some easy-to-check *sufficient* conditions;
- If the constraint can be written as $x[k] \succeq Mx[k]$, which is often the case, $M^*$ does not having $\perp$ entries is sufficient;

# Control characteristic equation

- **Definition**: the *control characteristic equation* $\mathcal{C}(\mathcal{R})$ associated to the problem $\mathcal{R}(A, B, E, D)$ is the following equation for the unknowns $\chi \in \mathcal{X}$, $\mu \in \mathcal{U}$ and $\lambda \in \mathbb{R}$

$$\begin{aligned} \lambda\chi &= A\chi \oplus B\mu; \\ E\chi &= D\chi. \end{aligned} \tag{6}$$

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- **Definition**: the *control characteristic spectrum* of a problem, $\Lambda(\mathcal{R})$, is the set of $\lambda$ such that $\{\lambda, \chi, \mu\}$ is a proper solution;
- $\Lambda(\mathcal{R})$ is a subset of the real line and contains the set of all the rates that are allowable to have for the system under control regime;

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- Intuition: controllable non-critical means that there is a solution for the problem that delays the system, even if just a little bit;
- It is more restrictive than being *controllable coupled*, but *all* problems found in literature thus far are both *controllable coupled* and *controllable non-critical*;

# Convergence number

- **Definition**: Given a square matrix $M$ with $\rho(M) \leq 0$, the *convergence number* $\kappa(M)$ is the smallest number $k$ such that

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- If $M$ has $n$ rows (and hence $n$ columns), then $\kappa(M) \leq n$.

# Main results

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- **Theorem 2**: a controllable coupled and controllable non-critical problem $\mathcal{R}$ is solvable *if and only if* its control characteristic equation $\mathcal{C}(\mathcal{R})$ has a proper solution $\{\lambda, \chi, \mu\}$. The control action is a simple state feedback of the form

$$u[k] = Fx[k] \tag{8}$$

in which $F = \mu(-\chi)^T$. Furthermore, the closed loop system will have eigenvalue equal to $\lambda$ and convergence to $\mathcal{S}$ is achieved in at most $\kappa(\lambda^{-1}A)$ events.

# Solving the control characteristic equation

- The control characteristic equation $\mathcal{C}(\mathcal{R})$ can be written conveniently as

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- *Pseudopolynomial algorithms*: not very difficult to solve currently for medium-sized systems ( $<100$ unknowns);
- Another technique to solve it efficiently for large systems (near to a thousand of unknowns) is under development by the author and his collaborators.

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- In the context of the control characteristic equation $\mathcal{C}(\mathcal{R})$, $y = (\chi^T \ \mu^T)^T$, but $y \neq \perp$ does not guarantee, in principle, that $\chi$ has not $\perp$ entries. That is, it does not guarantee that the solution generated from $y$ will be proper to $\mathcal{C}(\mathcal{R})$;

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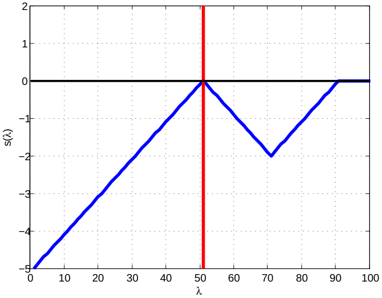
- If the problem is controllable coupled, however, any solution to the two-sided eigenproblem generates a proper solution to the control characteristic equation $\mathcal{C}(\mathcal{R})$, that is, $y \neq \perp$ implies that $\chi$ does not have $\perp$ entries;

# Solving the control characteristic equation

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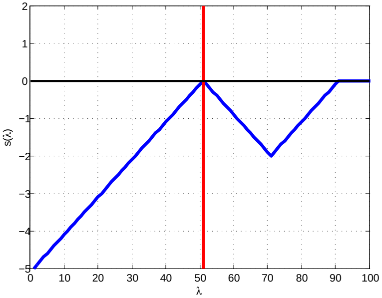


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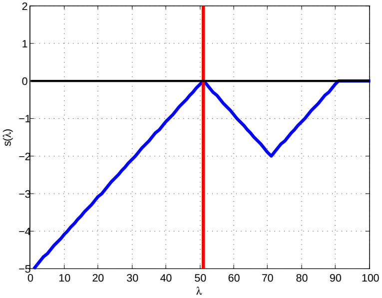


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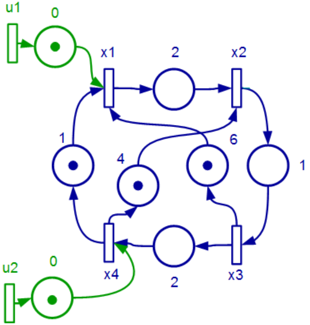
- This can be done by *solving* the associated mean-payoff game at the point $\lambda$;
- In the given example, $\rho(A) = 50$ (red line). The control characteristic spectra is $\Lambda = \{50\} \cup [90, 100]$, which is not the singleton $\{\rho(A)\} = \{50\}$, so the problem is controllable non-critical;

# Example of problem

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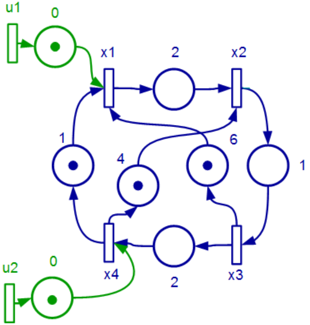

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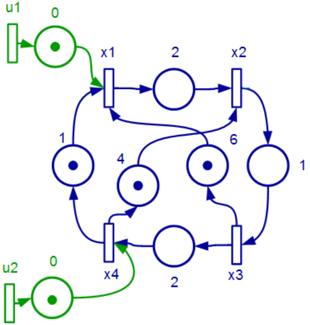


- Problem of controlling a cluster tool in a waffer manufacturing process;
- 4 states, 2 control inputs;
- Equation

$$x[k+1] = Ax[k] \oplus Bu[k];$$
$$A = \begin{pmatrix} \perp & \perp & 6 & 1 \\ \perp & \perp & 8 & 4 \\ \perp & \perp & 9 & 5 \\ \perp & \perp & 11 & 7 \end{pmatrix};$$
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- Constraints:

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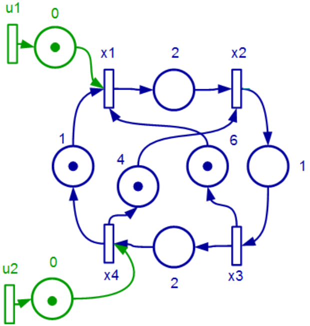

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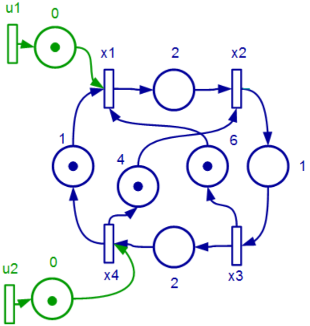


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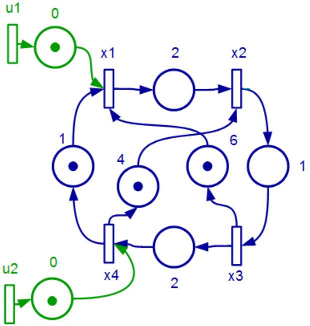


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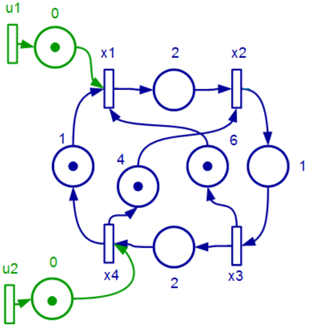


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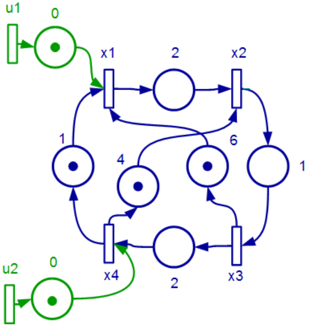


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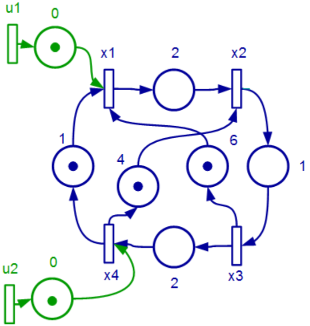


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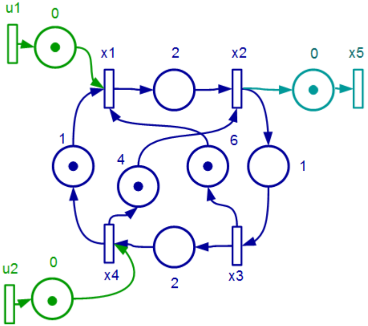


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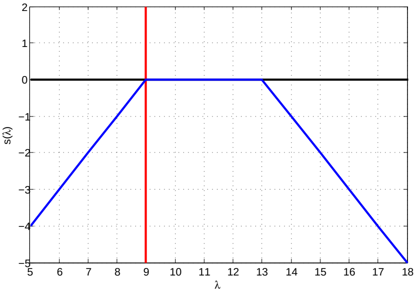

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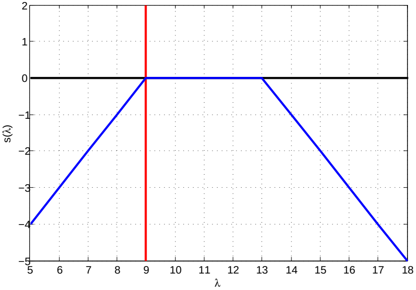


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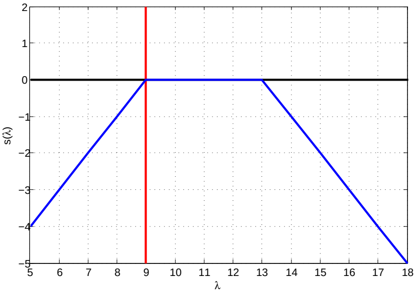


- Figure above shows that $\Lambda = [9, 13]$. There is an element other than $\rho(\hat{A}) = 9$, so the problem is *controllable non-critical*;
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- This means that the controller that will be derived will induce a periodic regime in which each event will fire at every $\lambda = 10$ time units. Furthermore, it will take at most $\kappa(\lambda^{-1}A) = 2$ events before this regime is achieved, whichever it is the initial condition;

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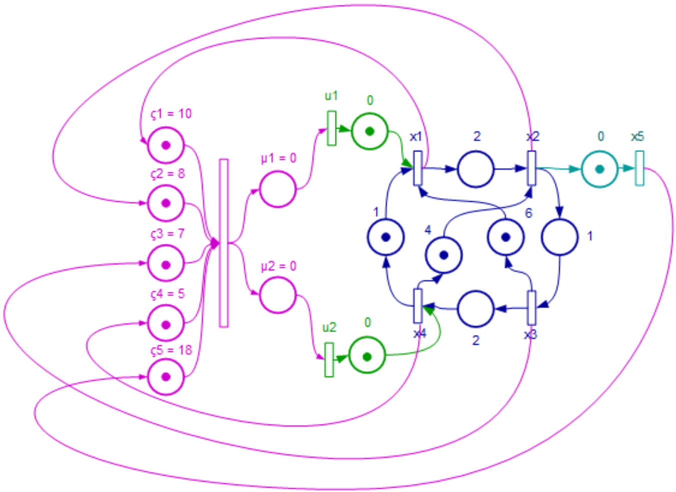

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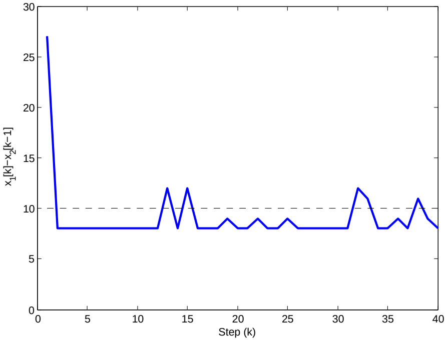

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- Interesting fact: in traditional linear system theory, observability conditions do not depend on the matrix $B$ (that connects inputs to states). Due to the absence of subtraction, in the tropical setting *observability do depend on the input matrix $B$*!

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  for the unknowns $\{\chi_1, \chi_2, \mu_1, \mu_2, \lambda\}$;
- **Conjecture**: the regulation problem $\mathcal{R}(A, B, E, D)$ has a solution if and only if there exists a $K$ such that $\mathcal{C}_K(\mathcal{R})$ has a proper solution;

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- This implies that critical problems demand control topologies which are more complex, maybe $u[k+1] = Gu[k] \oplus Fx[k]$;

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  in which $g$ is topical in $x[k]$ and $u[k]$;
- Let $L(x)$ and $R(x)$ be topical functions;
- **Generalized regulation problem**: derive the control law $u[k]$ such that the trajectory $x[k]$ of a topical system converges and stays to the set described by $\{x \mid L(x) = R(x)\}$;

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- Unfortunately, particular cases of it were proven to be *NP-hard*;

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- More results/details in V. M. Gonçalves's thesis [5].

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