# Capture Basin Approximation using Interval Analysis

Mehdi Lhommeau, Luc Jaulin and Laurent Hardouin

*LISA - Université d'Angers, France*
*LDTN - ENSIETA, France*

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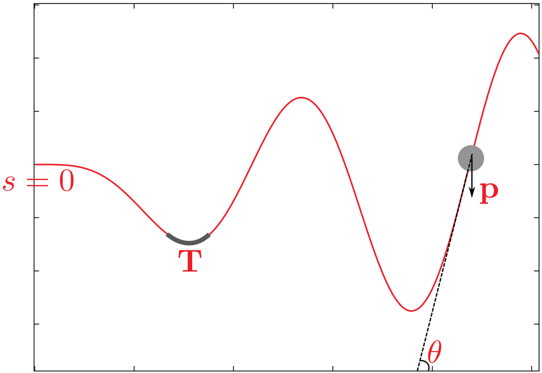


- Curvilinear coordinates :
  - $s$ is the position on the track, measured by the path length
  - $\dot{s}$ is the velocity of the ball

# Nonlinear Dynamical System

## System

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \quad (1)$$

Where $t > 0$ :

- $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector
- $\mathbf{u}(t) \in \mathbf{U} \subset \mathbb{R}^m$ is the control

## Roller Coaster dynamic

If $\mathbf{x} = (s, \dot{s})$ then

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g \sin(\Theta(x_1)) - \alpha x_2 + u \end{cases}$$

Where

- $\alpha$ : friction force
- $\Theta : x_1 \mapsto \theta$ is a given function
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The whole trajectory is given by

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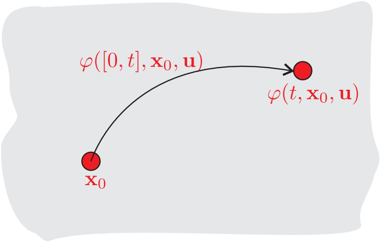

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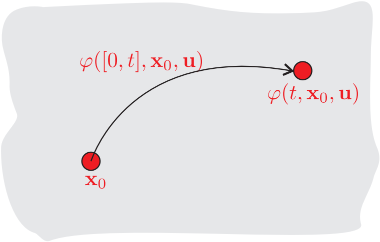

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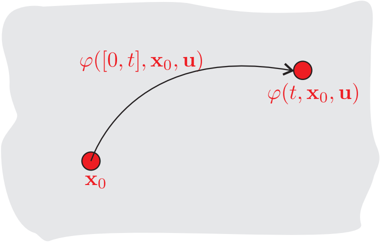

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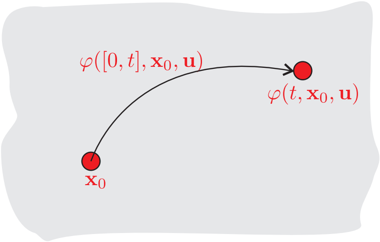
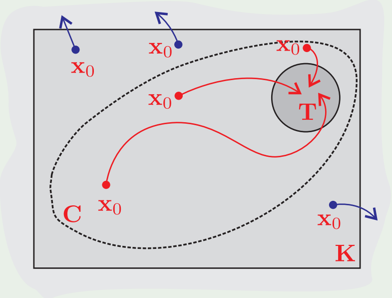


## Definition

Define two compact sets **T** and **K** such that $\mathbf{T} \subset \mathbf{K} \subset \mathbb{R}^n$. **T** is the target and **K** is the viable set. The *capture basin* **C** is the subset of states of **K** from which there exists at least one solution inside **K** reaching the target **T** in finite time $t$ :

$$\mathbf{C} \quad = \quad \{\mathbf{x}_0 \in \mathbf{K} \mid \exists t > 0, \exists \mathbf{u} \in \mathcal{U}, \varphi(t; \mathbf{x}_0, \mathbf{u}) \in \mathbf{T} \text{ and } \varphi([0,t]; \mathbf{x}_0, \mathbf{u}) \subset \mathbf{K}\}.$$



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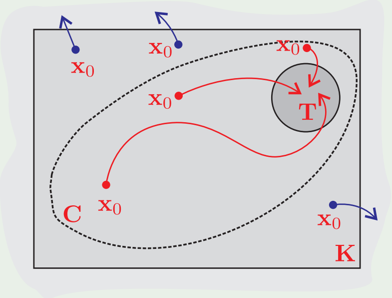
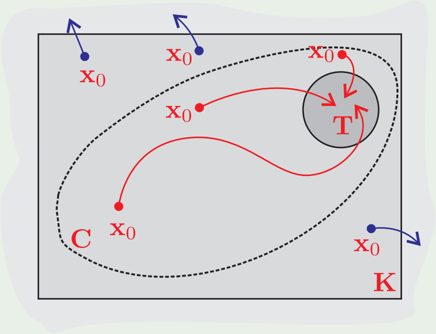


## Guaranteed Approximation

Find two sets $\mathbf{C}^-$ and $\mathbf{C}^+$ such that

$$\mathbf{C}^- \subset \mathbf{C} \subset \mathbf{C}^+$$



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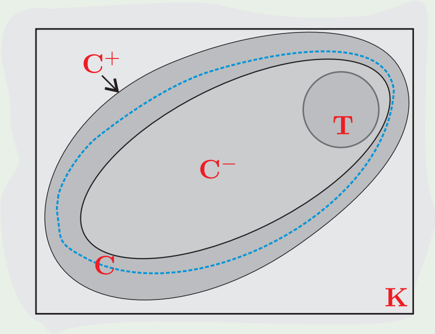

## Proposition 1

We have

(i) $\mathbf{x}_0 \in \mathbf{T} \Rightarrow \mathbf{x}_0 \in \mathbf{C}$

(ii) $\mathbf{x}_0 \notin \mathbf{K} \Rightarrow \mathbf{x}_0 \notin \mathbf{C}$

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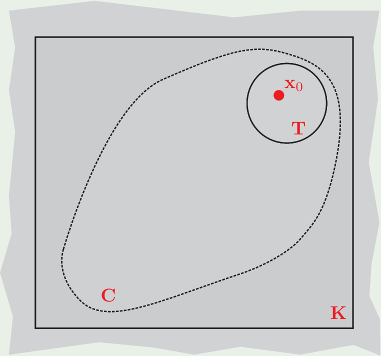

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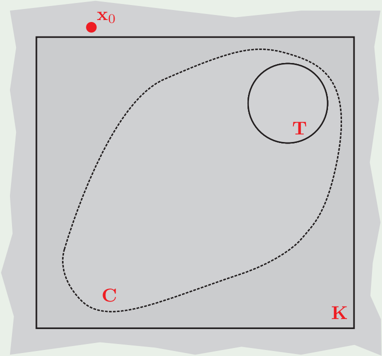

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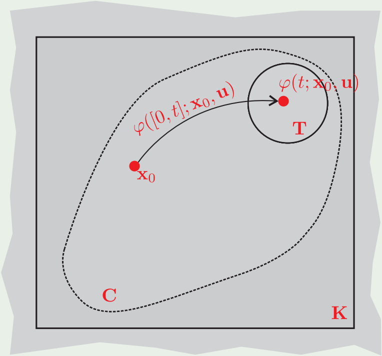

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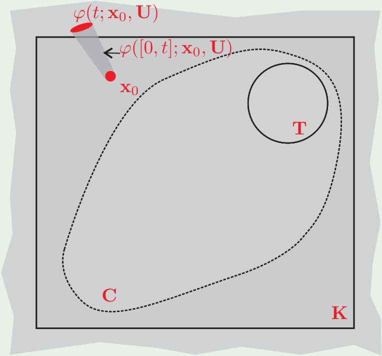
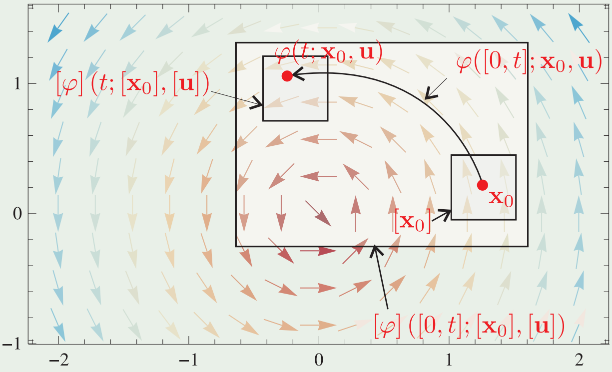


## Notation

If $[t] \in \mathbb{IR}$, $[\mathbf{x}_0] \in \mathbb{IR}^n$ and $[\mathbf{u}] \in \mathbb{IR}^m$

$$[\varphi]([t];[\mathbf{x}_0],[\mathbf{u}]) \quad := \quad \{\varphi(t;\mathbf{x}_0,\mathbf{u}) \mid t \in [t], \mathbf{x}_0 \in [\mathbf{x}_0], \mathbf{u} \in \mathcal{U}\}.$$



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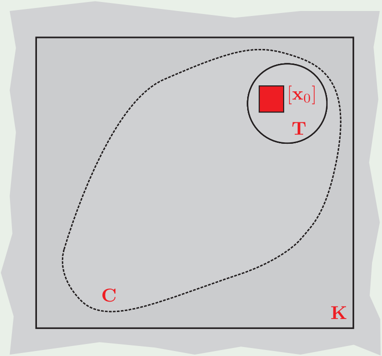

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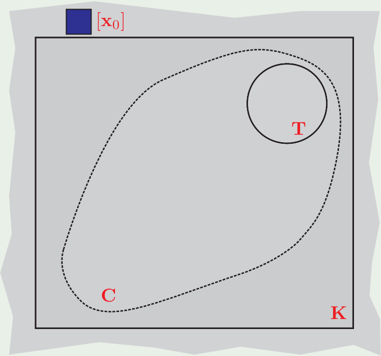

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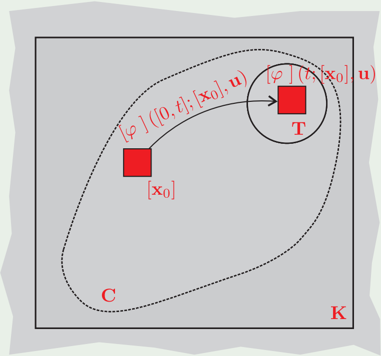

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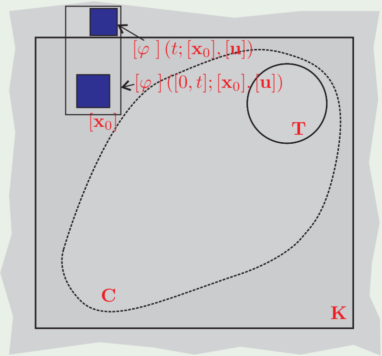
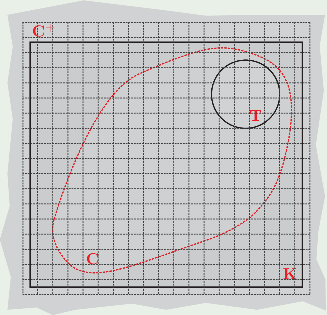


## Algorithm

- Initialize the sets $\mathbf{C}^- = \emptyset$ and $\mathbf{C}^+$ is a union of boxes covering $\mathbf{K}$




- Iterate :

  1 Take a box $[\mathbf{x}_0]$ in $\mathbf{C}^+$
  2 If $[\mathbf{x}_0] \subset \mathbf{T}$ then $\mathbf{C}^- := \mathbf{C}^- \cup [\mathbf{x}_0]$ ; goto 1 ;
  3 If $[\mathbf{x}_0] \cap \mathbf{K} = \emptyset$ then $\mathbf{C}^+ := \mathbf{C}^+ \backslash [\mathbf{x}_0]$ ; goto 1 ;
  4 take $t \in \mathbb{R}^+$ and $\mathbf{u} \in [\mathbf{u}]$ ;
  5 If $[\varphi](t; [\mathbf{x}_0], \mathbf{u}) \subset \mathbf{C}^-$ and $[\varphi]([0, t]; [\mathbf{x}_0], \mathbf{u}) \subset \mathbf{K}$
  then $\mathbf{C}^- := \mathbf{C}^- \cup [\mathbf{x}_0]$ ; goto 1 ;
  6 If $[\varphi](t; [\mathbf{x}_0], [\mathbf{u}]) \cap \mathbf{C}^+ = \emptyset$ and $[\varphi]([0, t]; [\mathbf{x}_0], [\mathbf{u}]) \cap \mathbf{T} = \emptyset$
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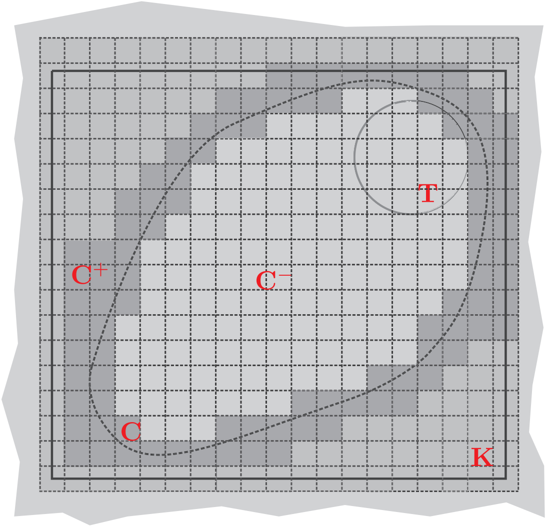
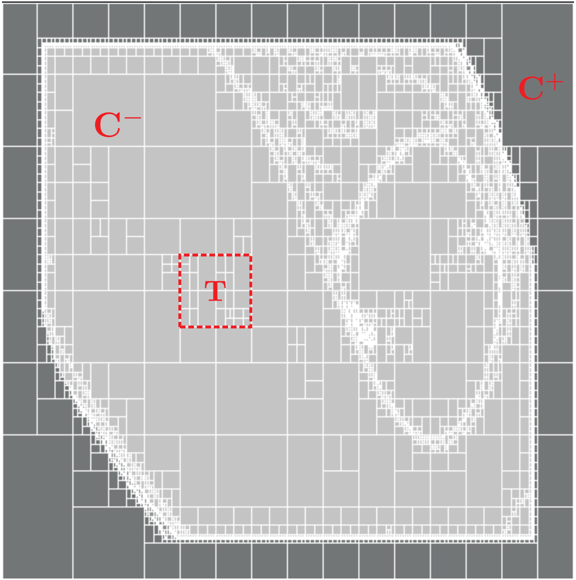
- Until no more change can be observed


T
C+
C−
C
K

## Roller Coaster



# Conclusion

- We have presented a new algorithm that provides guaranteed inner and outer approximations of the capture basin
  - Hybrid systems
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- Inner and outer approximations of the kernel viability

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