

Capture Basin Approximation using Interval Analysis

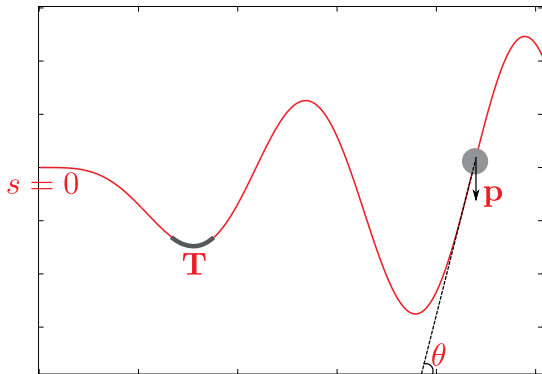
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Octobre 2009

Problem statement



- Curvilinear coordinates :

- ▶ s is the position on the track, measured by the path length
- ▶ \dot{s} is the velocity of the ball

System

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \quad (1)$$

Where $t > 0$:

- $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector
- $\mathbf{u}(t) \in \mathbf{U} \subset \mathbb{R}^m$ is the control

Roller Coaster dynamic

If $\mathbf{x} = (s, \dot{s})$ then

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -g \sin(\Theta(x_1)) - \alpha x_2 + u \end{cases}$$

Where

- α : friction force
- $\Theta : x_1 \mapsto \theta$ is a given function
- u is the control

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Flow

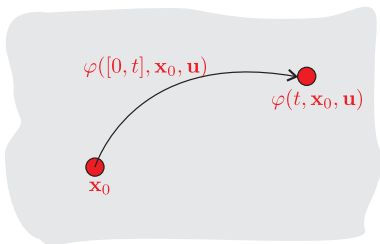
$$t \mapsto \mathbf{x}(t) = \varphi(t; \mathbf{x}_0, \mathbf{u}), \quad (2)$$

is the unique solution to (1). Where

- $\mathbf{x}(0) = \mathbf{x}_0$ is the initial condition
- $\mathbf{u} \in \mathcal{U} := \{\mathbf{u} : [0, t] \rightarrow \mathbf{U} \mid t \geq 0 \text{ and } \mathbf{u} \text{ is piecewise continuous}\}$ is the control function

The whole trajectory is given by

$$\varphi([0, t]; \mathbf{x}_0, \mathbf{u}) = \bigcup_{\tau \in [0, t]} \varphi(\tau; \mathbf{x}_0, \mathbf{u}). \quad (3)$$



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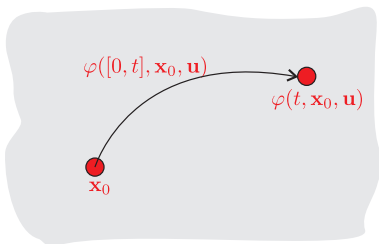
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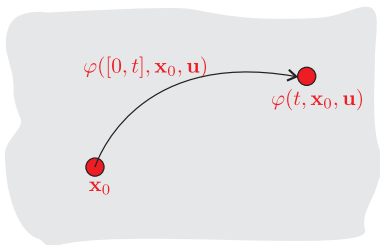
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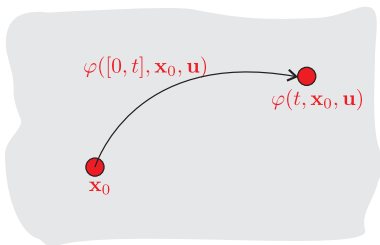
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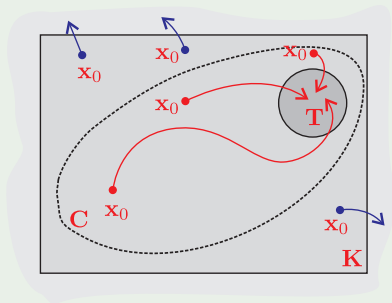
$$\varphi([0, t]; \mathbf{x}_0, \mathbf{u}) = \bigcup_{\tau \in [0, t]} \varphi(\tau; \mathbf{x}_0, \mathbf{u}). \quad (3)$$



Definition

Define two compact sets \mathbf{T} and \mathbf{K} such that $\mathbf{T} \subset \mathbf{K} \subset \mathbb{R}^n$. \mathbf{T} is the target and \mathbf{K} is the viable set. The *capture basin* \mathbf{C} is the subset of states of \mathbf{K} from which there exists at least one solution inside \mathbf{K} reaching the target \mathbf{T} in finite time t :

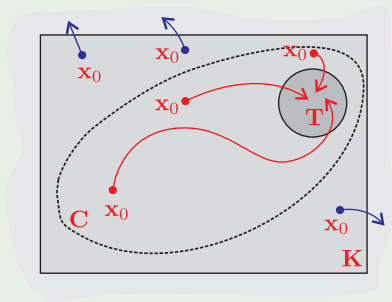
$$\mathbf{C} = \{x_0 \in \mathbf{K} \mid \exists t > 0, \exists u \in \mathcal{U}, \varphi(t; x_0, u) \in \mathbf{T} \text{ and } \varphi([0, t]; x_0, u) \subset \mathbf{K}\}.$$



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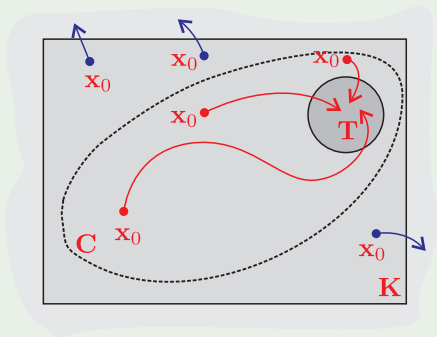
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Guaranteed Approximation

Find two sets C^- and C^+ such that

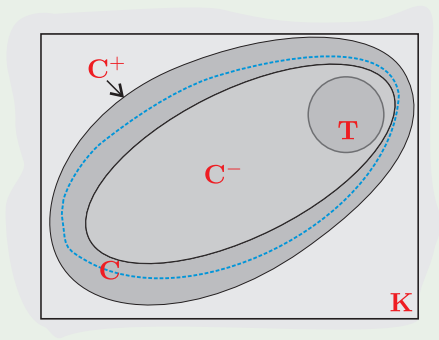
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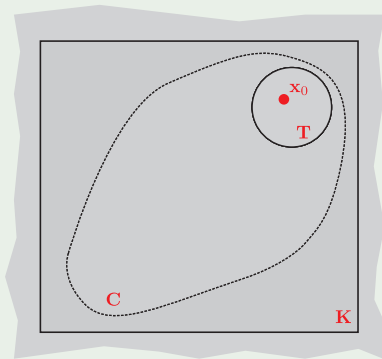
We have

(i) $x_0 \in T \Rightarrow x_0 \in C$

(ii) $x_0 \notin K \Rightarrow x_0 \notin C$

(iii) $(\exists t \geq 0, \exists u \in \mathcal{U}, \varphi(t; x_0, u) \in C \wedge \varphi([0, t]; x_0, u) \subset K) \Rightarrow x_0 \in C$

(iv) $(\exists t \geq 0, \forall u \in \mathcal{U}, \varphi(t; x_0, u) \notin C \wedge \varphi([0, t]; x_0, u) \cap T = \emptyset) \Rightarrow x_0 \notin C$



Proposition 1

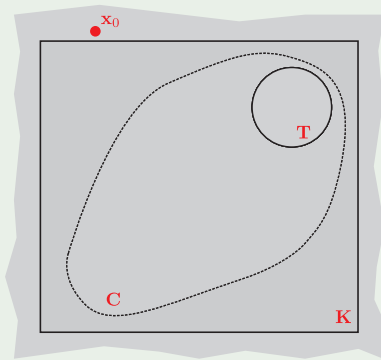
We have

(i) $\mathbf{x}_0 \in \mathbf{T} \Rightarrow \mathbf{x}_0 \in \mathbf{C}$

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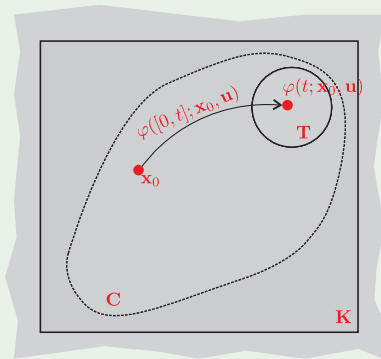
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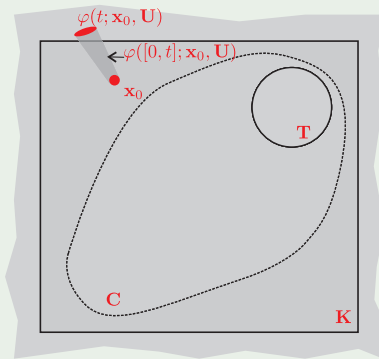
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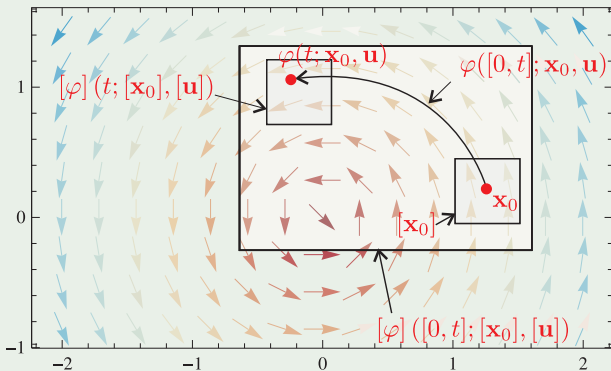
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Notation

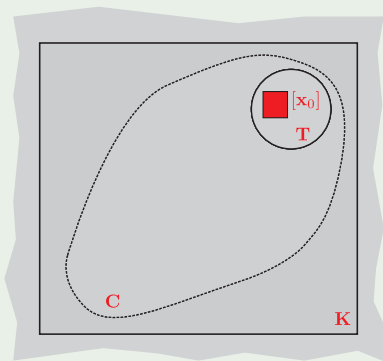
If $[t] \in \mathbb{IR}$, $[\mathbf{x}_0] \in \mathbb{IR}^n$ and $[\mathbf{u}] \in \mathbb{IR}^m$

$$[\varphi]([t]; [\mathbf{x}_0], [\mathbf{u}]) := \{\varphi(t; \mathbf{x}_0, \mathbf{u}) \mid t \in [t], \mathbf{x}_0 \in [\mathbf{x}_0], \mathbf{u} \in \mathcal{U}\}.$$



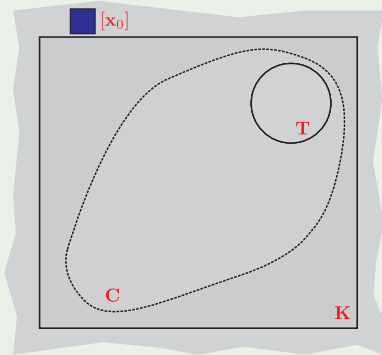
Proposition 2

- (i) $[x_0] \subset \mathbf{T} \Rightarrow [x_0] \subset \mathbf{C}$
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- (iii) $(\exists \mathbf{u} \in [\mathbf{u}], [\varphi](t; [x_0], \mathbf{u}) \subset \mathbf{C} \wedge ([\varphi]([0, t]; [x_0], \mathbf{u})) \subset \mathbf{K}) \Rightarrow [x_0] \subset \mathbf{C}$
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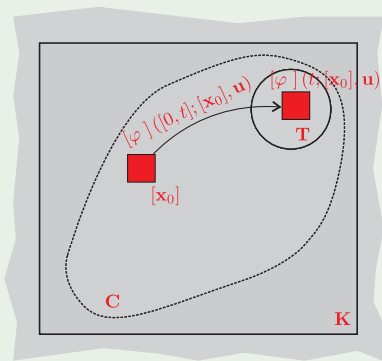
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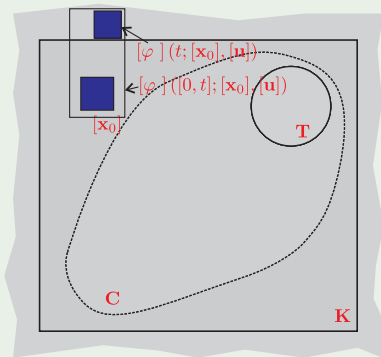
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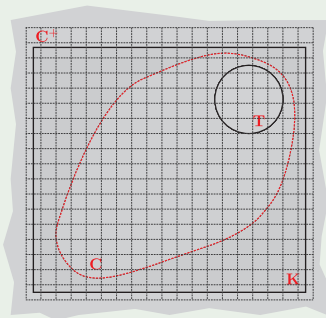


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- Initialize the sets $\mathbf{C}^- = \emptyset$ and \mathbf{C}^+ is a union of boxes covering \mathbf{K}

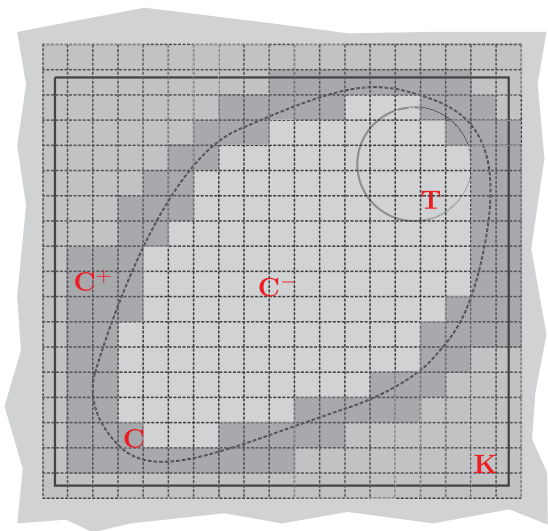


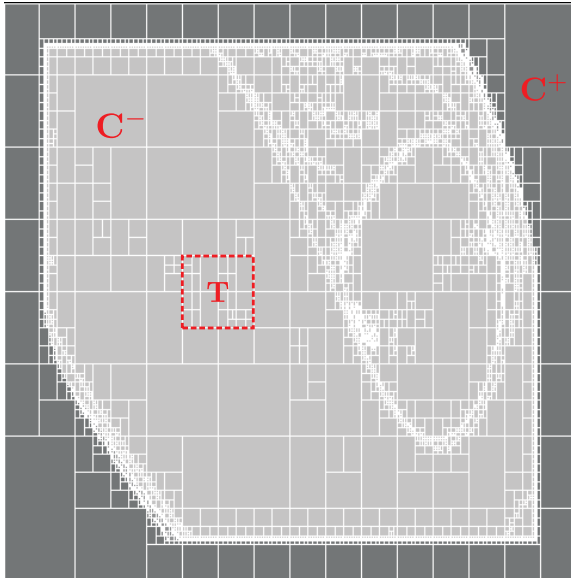
- Iterate :

- 1 Take a box $[\mathbf{x}_0]$ in \mathbf{C}^+
- 2 If $[\mathbf{x}_0] \subset \mathbf{T}$ then $\mathbf{C}^- := \mathbf{C}^- \cup [\mathbf{x}_0]$; goto 1;
- 3 If $[\mathbf{x}_0] \cap \mathbf{K} = \emptyset$ then $\mathbf{C}^+ := \mathbf{C}^+ \setminus [\mathbf{x}_0]$; goto 1;
- 4 take $t \in \mathbb{R}^+$ and $\mathbf{u} \in [\mathbf{u}]$;
- 5 If $[\varphi](t; [\mathbf{x}_0], \mathbf{u}) \subset \mathbf{C}^-$ and $[\varphi]([0, t]; [\mathbf{x}_0], \mathbf{u}) \subset \mathbf{K}$
then $\mathbf{C}^- := \mathbf{C}^- \cup [\mathbf{x}_0]$; goto 1;
- 6 If $[\varphi](t; [\mathbf{x}_0], [\mathbf{u}]) \cap \mathbf{C}^+ = \emptyset$ and $[\varphi]([0, t]; [\mathbf{x}_0], [\mathbf{u}]) \cap \mathbf{T} = \emptyset$
then $\mathbf{C}^+ := \mathbf{C}^+ \setminus [\mathbf{x}_0]$; goto 1;

- Until no more change can be observed

Output of the algorithm





- We have presented a new algorithm that provides guaranteed inner and outer approximations of the capture basin
 - ▶ Hybrid systems
 - Roller coaster where the ball can jump off the track
- Inner and outer approximations of the kernel viability

$$\mathbf{V} = \{x_0 \in \mathbf{K} \mid \exists \mathbf{u} \in \mathcal{U}, \forall t > 0, \varphi(t; x_0, \mathbf{u}) \in \mathbf{K}\}$$

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