Building of an optimal triangular contractor

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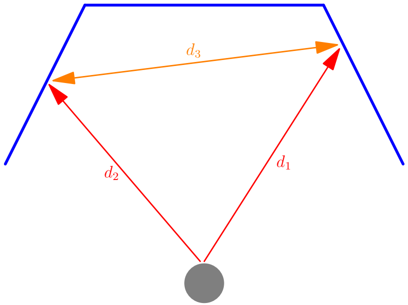
January 26, 2024

Plan

Problem Presentation

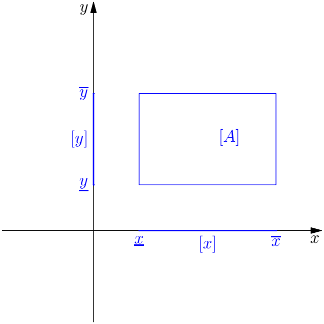
The aim of this study is to explore the localization of a mobile robot navigating within a known environment.



d_1 et d_2 represent the distances between the robot and the two obstacles, and d_3 represents the distance separating the two obstacles.

Problem Presentation

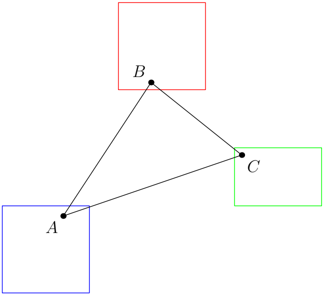
We denote the position of the robot by (x, y). The unknown position of the robot is assumed to belong to a box represented by intervals $[x]$ and $[y]$.



Similarly, the positions of the obstacles are represented by intervals.

Geometric interpretation

Let A represent the position of the robot, and B and C represent the positions of the two obstacles. Consider $[A]$, $[B]$, and $[C]$ as the intervals vector (box) within which the positions of A, B, and C are asumed belong to



Geometric interpretation

The objective is to minimize the boxes $[A]$, $[B]$, and $[C]$, and therefore the intervals representing them, while remaining consistent with the constraint ABC being a triangle with sides d_1, d_2, and d_3.

Problem Modeling

Let $\mathcal{P}$ be a Cartesian plane, and A and B be two points in $\mathcal{P}$ with coordinates (x_A, y_A) and (x_B, y_B). the distance between A and B is

$$AB := \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}.$$

Let d be a positive real number. Consider the following equation:

$$\left\{ \begin{array}{rcl} (x_A - x_B)^2 + (y_A - y_B)^2 - d_1^2 & = & 0, \\ (x_A, y_A) & \in [A], & \\ (x_B, y_B) & \in [B] & \end{array} \right. \tag{1}$$

The solution to the above equation can be viewed as the set of points $(A, B) \in [A] \times [B]$ such that $AB = d_1$.

Problem Modeling

Let's consider the following system of equations :

$$\begin{cases} (x_A - x_B)^2 + (y_A - y_B)^2 - d_1^2 = 0 \\ (x_A - x_C)^2 + (y_A - y_C)^2 - d_2^2 = 0 \\ (x_B - x_C)^2 + (y_B - y_C)^2 - d_3^2 = 0 \end{cases} \tag{2}$$

with

$$\begin{cases} (x_A, y_A) \in [A] = [x_A] \times [y_A] \\ (x_B, y_B) \in [B] = [x_B] \times [y_B] \\ (x_C, y_C) \in [C] = [x_C] \times [y_C] \end{cases}$$

Problem Modeling

Equation (2) can be expressed in vector form as follows

$$\mathcal{H}_{ABC} : \begin{cases} \mathrm{f}(\mathrm{x}) & = 0, \\ \mathrm{x} & \in [\mathrm{x}] \end{cases} \tag{3}$$

where

$$[\mathrm{x}] = (([x_A] \times [y_A]), ([x_B] \times [y_B]), ([x_C] \times [y_C]))$$

and f is the vector function defined as

$$\begin{cases} f_1(\mathrm{x}) = (x_A - x_B)^2 + (y_A - y_B)^2 - d_1^2 \\ f_2(\mathrm{x}) = (x_A - x_C)^2 + (y_A - y_C)^2 - d_2^2 \\ f_3(\mathrm{x}) = (x_B - x_C)^2 + (y_B - y_C)^2 - d_3^2 \end{cases}$$

(3) is also called a Constraint Satisfaction Problem..

Problem Modeling

We denote by

$$\mathbb{S}_j := \{\mathrm{x} \in [\mathrm{x}] : \ \mathrm{f}_j(\mathrm{x}) = 0\}$$

the solution of the j^{th} constraint and by

$$\mathbb{S} := \{\mathrm{x} \in [\mathrm{x}] : \ \mathrm{f}(\mathrm{x}) = 0\}$$

the solution of the system.
If $\mathrm{x} \in \mathbb{S}$, then

$$AB = d_1, \ AC = d_2 \ \text{ and } \ BC = d_3.$$

Therefore, $\mathbb{S}$ represents the set of coordinates for triangles with sides $AB = d_1$, $AC = d_2$, and $BC = d_3$..

Problem Modeling

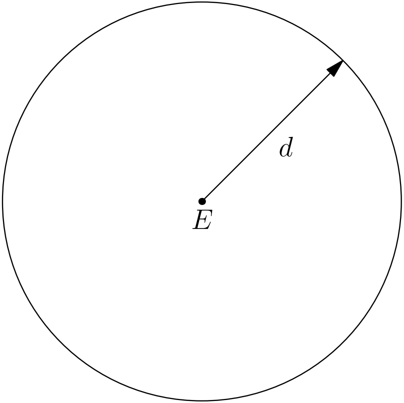
The objective of this work is to develop a method to compute the small subset $[\mathrm{x}']$ of $[x]$

$$\mathbb{S} \subseteq [\mathrm{x}'].$$

In other words, the goal is to approximate the set $\mathrm{f}^{-1}(\{0\})$.

Geometric construction

Let $\mathcal{P}$ be a Cartesian plane, E a point in $\mathcal{P}$ with coordinates (x_E, y_E), and d a positive real number. The set $S_d(E)$ of points in $\mathcal{P}$ which are at a distance d from the point E forms the circle $\mathcal{C}$ with center E and radius d.



Geometric construction of the solution

Let d be a positive real number. The set of points that are at a distance of d from at least one point in $[A]$ is defined by

$$S_d([A]) := \bigcup_{A \in [A]} S_d(A).$$

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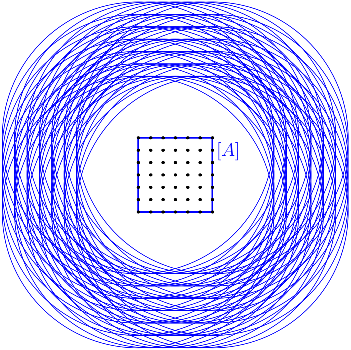

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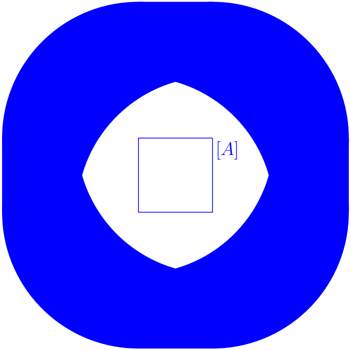


Geometric construction of the solution

Thus,

- $\mathbb{S}_1 := (S_d([A]) \cap [B]) \cup (S_d([B]) \cap [A])$;
- $\mathbb{S}_2 := (S_d([A]) \cap [C]) \cup (S_d([C]) \cap [A])$;
- $\mathbb{S}_3 := (S_d([B]) \cap [C]) \cup (S_d([C]) \cap [B])$.

Geometric construction of the solution

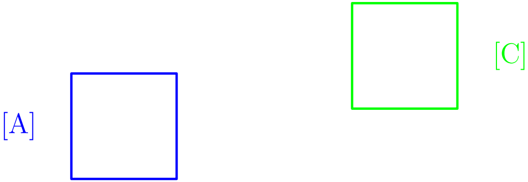
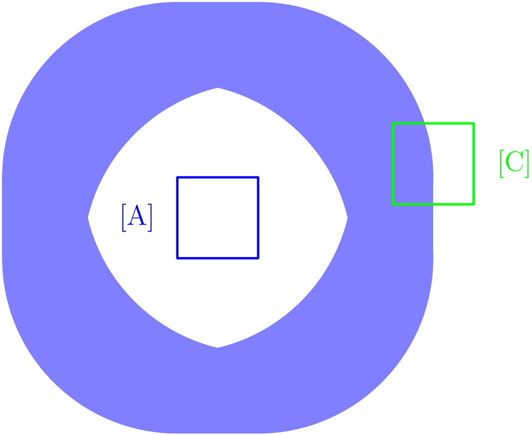


Figure: Illustration of the solution $\mathbb{S}_2$.

Geometric construction of the solution



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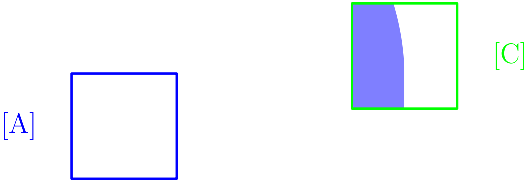

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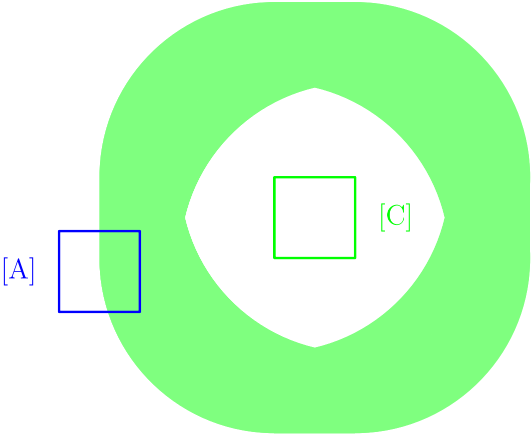

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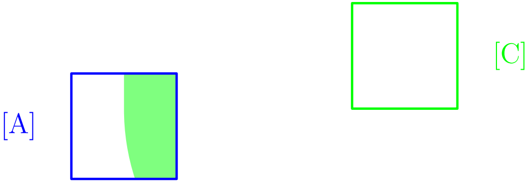

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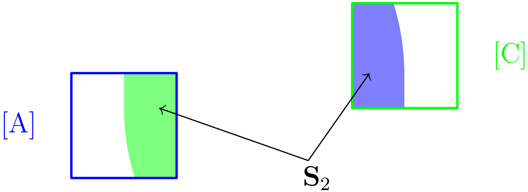


Figure: Illustration of the solution $\mathbb{S}_2$.

Contractor

Definition

A contactor of $\mathbb{S}$ is a function defined by

$$\begin{aligned} \mathcal{C}_{\mathbb{S}} : \mathbb{IR}^n &\rightarrow \mathbb{IR}^n \\ [\mathrm{x}] &\mapsto \mathcal{C}_{\mathbb{S}}([\mathrm{x}]) \end{aligned}$$

which satisfy

$$\forall\ [\mathrm{x}] \in \mathbb{IR}^n, \begin{cases} \mathcal{C}_{\mathbb{S}}([\mathrm{x}]) & \subseteq [\mathrm{x}] \\ \mathcal{C}_{\mathbb{S}}([\mathrm{x}]) \cap \mathbb{S} & = \mathbb{S} \cap [\mathrm{x}] \end{cases} \tag{4}$$

Contractor

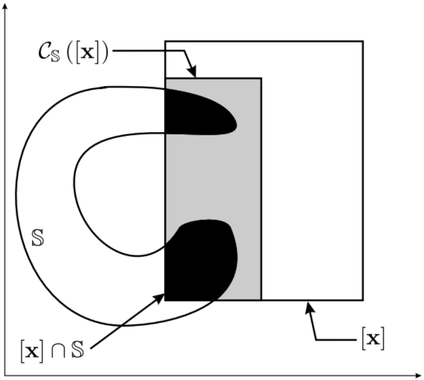


Figure: Illustration of the contraction of $\mathbb{S} \cap [x]$.

Projection

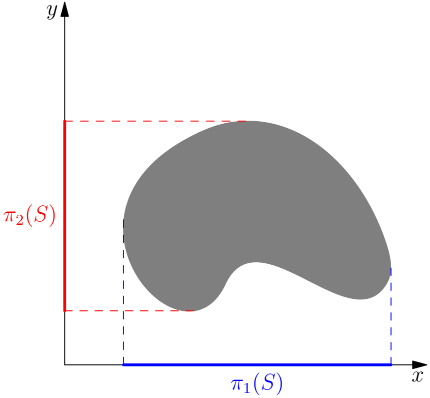
Let $\mathrm{J} = (j_1, \ldots, j_m)$ be an index set. We denote by xJ the vector $(xj_1, \ldots, x_{j_m})$.

Definition: Projection

Let $\mathrm{J} = (1, \ldots, i-1, i+1, \ldots, n)$ be a subset of $(1, \ldots, n)$. The projection of the solution $\mathbb{S}_j$ onto the x_i axis is defined by

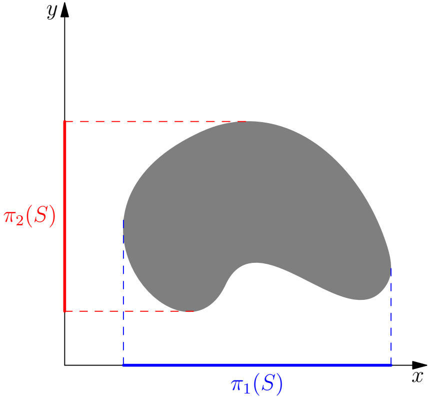
$$\pi_i(\mathbb{S}_j) := \{x_i \in [x_i] \mid \exists \mathrm{x_J} \in [\mathrm{x_J}] : \; f_j(\mathrm{x}) = 0\} \,. \tag{5}$$

- If $x_i \in \pi_i(\mathbb{S}_j)$, then the real number x_i is said to be consistent with $\mathbb{S}_j$.
- An interval $[x_i]$ is said to be consistent with respect to $\mathbb{S}_j$ if all its elements are consistent, i.e., $[x_i] \subseteq [\pi_i(\mathbb{S}_j)]$.


y
$\pi_2(S)$
x
$\pi_1(S)$

Projection

Look the smallest bounding box for the solution set $\mathbb{S}$ involves identifying the smallest intervals that encompass the projections $\pi_i(\mathbb{S})$ on each of the domains associated with the variables x_i.



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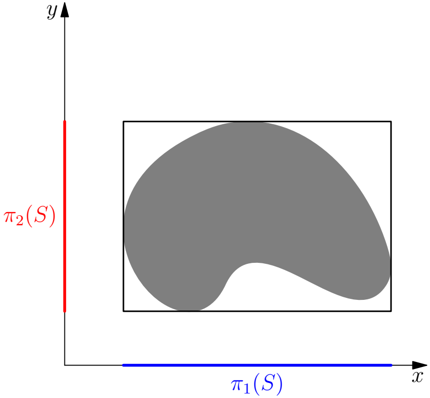


Triangle Contractor

Consider the constraint satisfaction problem $\mathcal{H}_{ABC}$ defined in (3).
Let $\mathcal{C}_{\mathbb{S}_1}$, $\mathcal{C}_{\mathbb{S}_2}$, $\mathcal{C}_{\mathbb{S}_3}$ be the contractors of $\mathbb{S}_1$, $\mathbb{S}_2$, and $\mathbb{S}_3$.
Let

$$\mathcal{C} := \mathcal{C}_{\mathbb{S}_1} \circ \mathcal{C}_{\mathbb{S}_2} \circ \mathcal{C}_{\mathbb{S}_3}.$$

A first contraction of $\mathcal{H}_{ABC}$ consists of successively applying the contractor $\mathcal{C}$ to $[x]$ until reaching on a fixed point.

Distance Contractor

Example

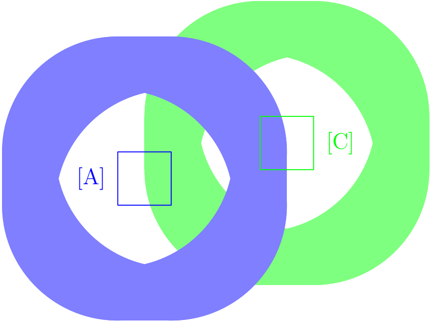
Apply the contractor $\mathcal{C}$ to the following system of equations

$$\left\{\begin{array}{rl} (x_A - x_B)^2 + (y_A - y_B)^2 - 11^2 & = 0 \\ (x_A - x_C)^2 + (y_A - y_C)^2 - 13^2 & = 0 \\ (x_B - x_C)^2 + (y_B - y_C)^2 - 9^2 & = 0 \end{array}\right. \tag{6}$$

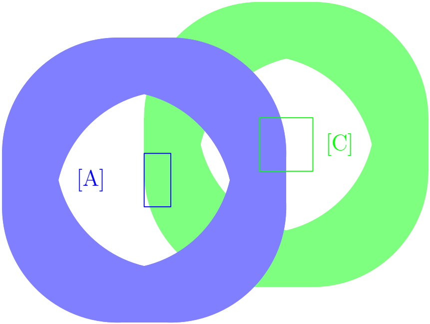
with

$$\left\{\begin{array}{rcl} (x_A, y_A) \in [A] & = & [-2, 4] \times [0, 6] \\ (x_B, y_B) \in [B] & = & [6, 12] \times [14, 20] \\ (x_C, y_C) \in [C] & = & [14, 20] \times [2, 10]. \end{array}\right.$$

Distance Contractor



Distance Contractor



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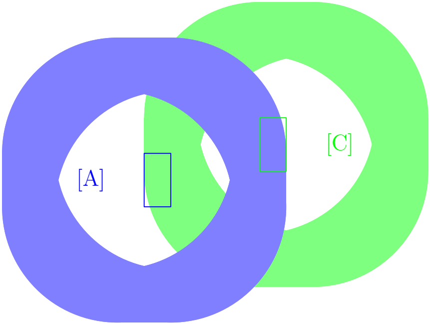

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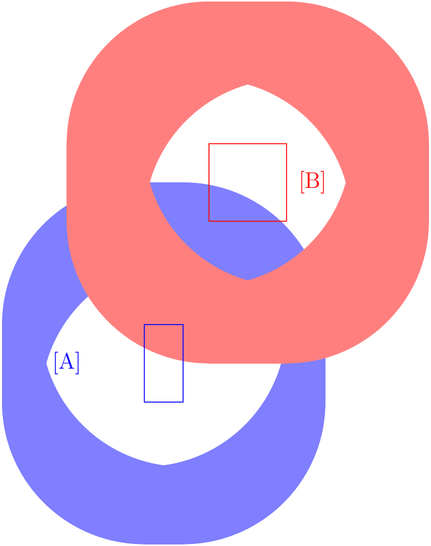

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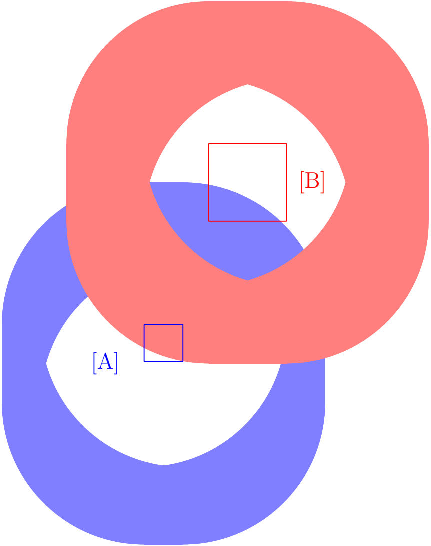

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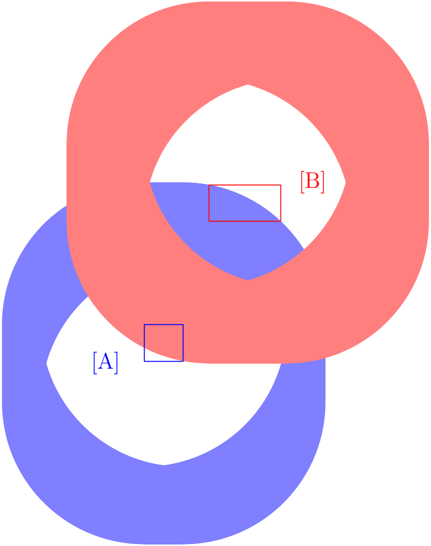

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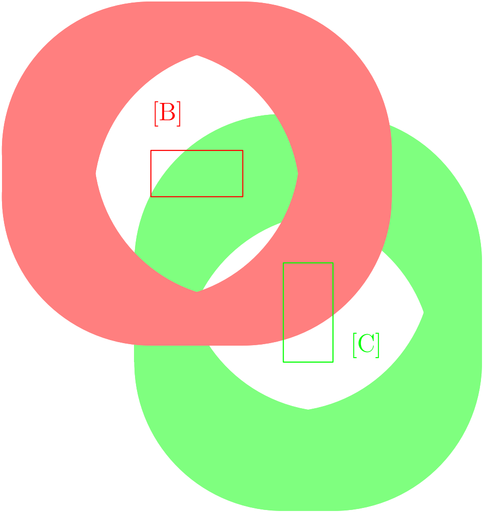

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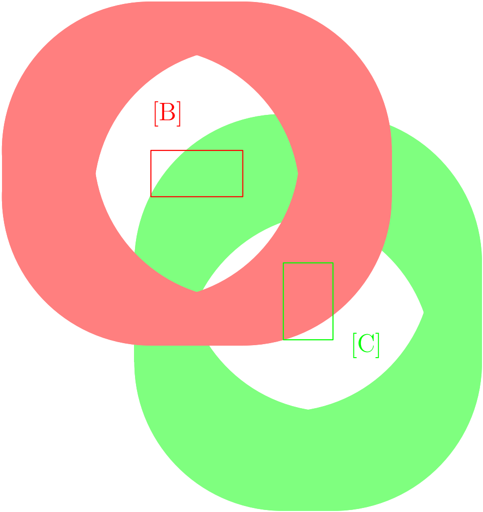

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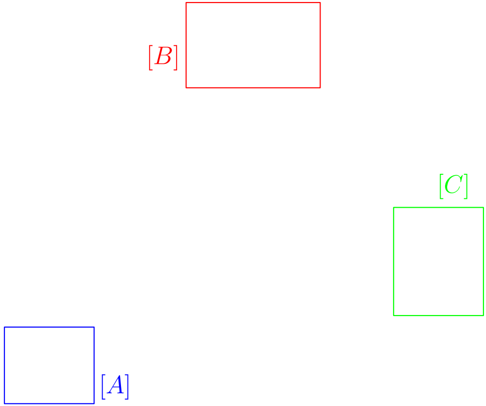


Distance Contractor

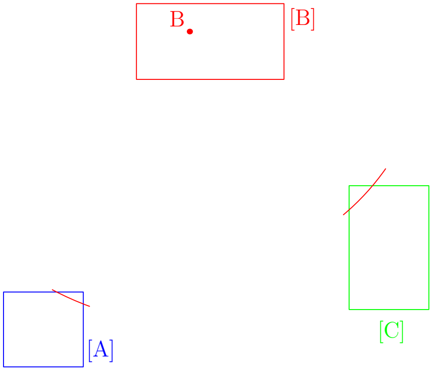
After contraction we obtain

- $[A] \;=\; [1,4] \times [3.18, 6]$,
- $[B] \;=\; [6, 11.55] \times [14, 16.82]$,
- $[C] \;=\; [14, 17] \times [5.34, 10]$.

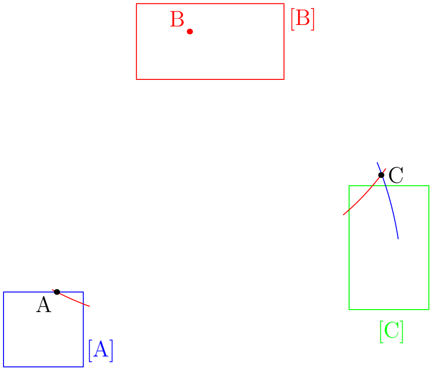
These boxes are such that

- The faces of $[A]$ intersect with $\mathbb{S}_1$ and $\mathbb{S}_2$,
- The faces of $[B]$ intersect with $\mathbb{S}_1$ and $\mathbb{S}_3$,
- The faces of $[C]$ intersect with $\mathbb{S}_2$ and $\mathbb{S}_3$.

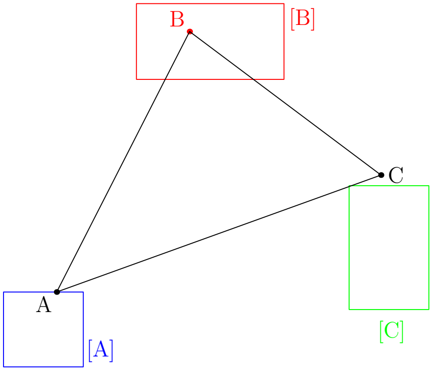
Triangle Contractor



Triangle Contractor



Triangle Contractor



Triangle Contractor

For any points A and A' in $\mathbb{S} \cap [A]$ with coordinates (x_A, y_A) and $(x_{A'}, y_{A'})$, we denote by

$$d(A, A') := \sqrt{(x_A - x_{A'})^2 + (y_A - y_{A'})^2}$$

the distance between A and A'.

Definition

Let $A \in \mathbb{S} \cap [A]$ and ε a positive real number. We call the open ball with center A and radius ε the set defined by

$$\mathbb{B}(A, \varepsilon) := \left\{A' \in \mathbb{S} \cap [A] : \ d(A, A') < \varepsilon\right\}.$$

Triangle Contractor

Definition

A point A is an interior point $\mathbb{S} \cap [A]$ if there exists a positive real number ε such that

$$\mathbb{B}(A, \varepsilon) \subset \mathbb{S} \cap [A].$$

From a geometric point of view, if a point is inside the solution, then there exists a space around the point that is solution space. In other words, if we move it (translation and rotation) in all directions in this space, we obtain a point that is a solution.

Triangle Contractor

Proposition

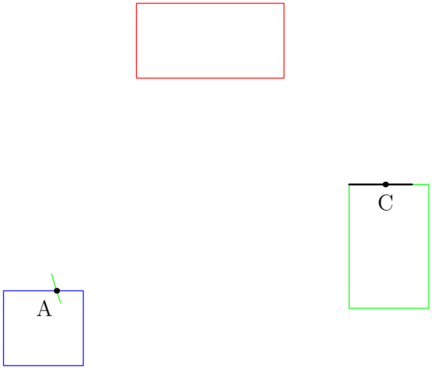
A triangle with coordinates $x \in \mathbb{S}$ is an interior triangle $\mathbb{S}$ if and only if its three vertices are is an interior pointe $\mathbb{S}$.

Triangle Contractor

Proposition

If two vertices of a triangle with coordinates $x \in \mathbb{S}$ are on the boundary of $\mathbb{S}$, then the third vertex is also on the boundary of $\mathbb{S}$.

Contracteur triangle



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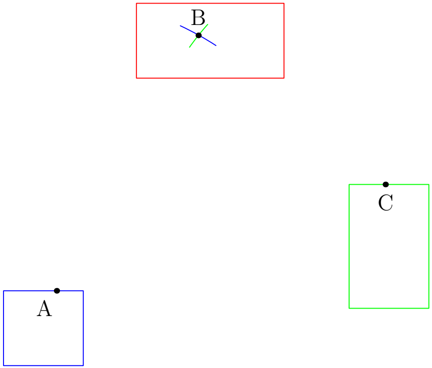

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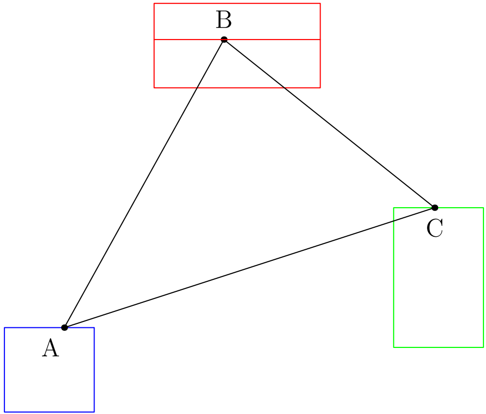


Triangle Contractor

Construction of the boundaries of $\mathbb{S}$

- Choose two parallel faces on two boxes,
- Construct a triangle ABC such that $AB = d_1$, $AC = d_2$, $BC = d_3$, and it has its two vertices on the chosen faces,
- If the third vertex is in the third box, then the two chosen faces are optimal boundaries of $\mathbb{S}$ and the third vertex is on an optimal boundary.
- Otherwise, one of the two faces does not belong to $\mathbb{S}$.

Triangle Contractor

Consider the system of equations from the previous example

$$\begin{cases} (x_A - x_B)^2 + (y_A - y_B)^2 - 121 = 0 & (C_1) \\ (x_A - x_C)^2 + (y_A - y_C)^2 - 169 = 0 & (C_2) \\ (x_B - x_C)^2 + (y_B - y_C)^2 - 81 = 0 & (C_3) \end{cases} \tag{7}$$

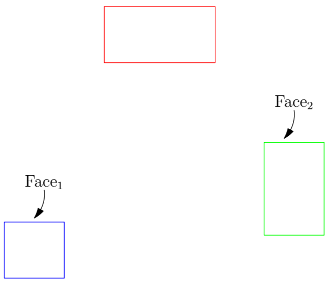
with

$$\begin{cases} (x_A, y_A) \in [A] = [1, 4] \times [3.18, 6] \\ (x_B, y_B) \in [B] = [6, 11.55] \times [14, 16.82] \\ (x_C, y_C) \in [C] = [14, 17] \times [5.34, 10]. \end{cases}$$

Triangle Contractor

- Choose the constraint C_2 and set $y_A = 6$ and $y_C = 10$. By Replacing the variables y_A and y_C in constraint C_2

$$(x_A - x_C)^2 - 153 \;=\; 0 \quad (C_2')$$



Triangle Contractor

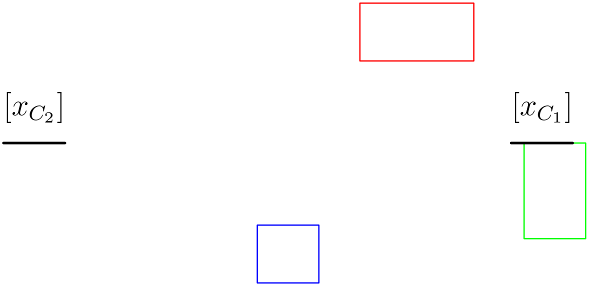
Let's find one solution of above equation (C_2') which represents the coordinates of points A and C of the two fixed faces such that $AC = 13$. For that

- Express x_C as a function of x_A,

$$x_{C_1} = x_A + \sqrt{153} \quad \text{or} \quad x_{C_2} = x_A - \sqrt{153}$$

Triangle Contractor

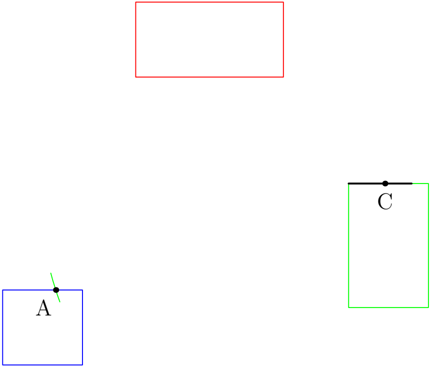
- choose the expression whose inclusion function intersects $[x_C]$,



So the solution expression is $x_C = x_A + \sqrt{153}$.

Triangle Contractor

- For $x_C = 15.369$, we have $x_A = 3$.



Triangle Contractor

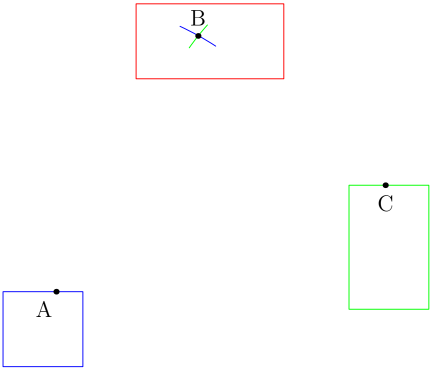
To obtain point B, replace the obtained variables (x_C, y_C) and (x_A, y_A) and solve the system:

$$\begin{cases} (3 - x_B)^2 + (6 - y_B)^2 - 121 & = 0 \quad (C_1') \\ (x_B - 15.369)^2 + (y_B - 10)^2 - 81 & = 0 \quad (C_3') \end{cases}$$

Triangle contractor

After solving, we obtain

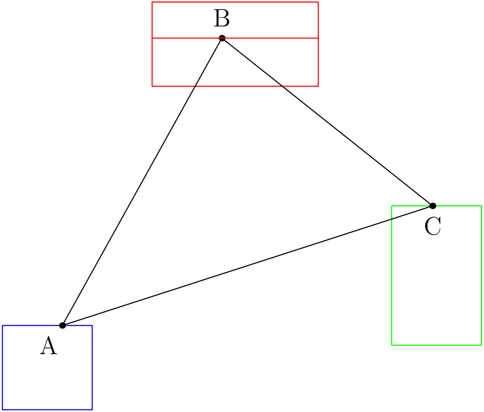
$$B = (8.33, 15.62).$$



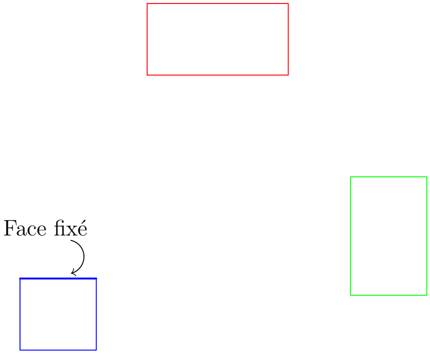
Contracteur triangle

Since $B \in [B]$, then

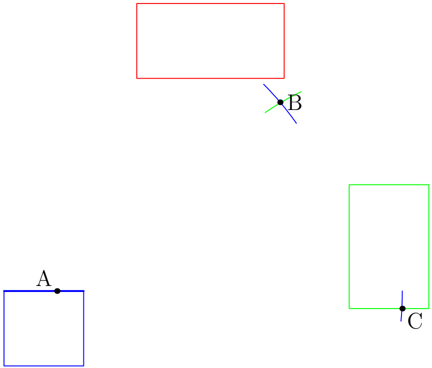
- $\overline{y_A} = 6$ is an optimal bound for $\mathbb{S} \cap [y_A]$,
- $\overline{y_C} = 10$ is an optimal bound for $\mathbb{S} \cap [y_C]$,
- B is on the border of $\mathbb{S} \cap [y_B]$.



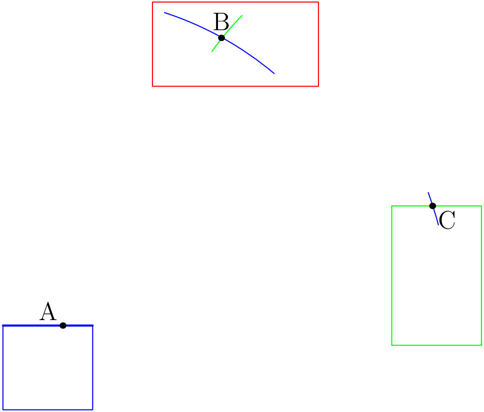
Contracteur triangle



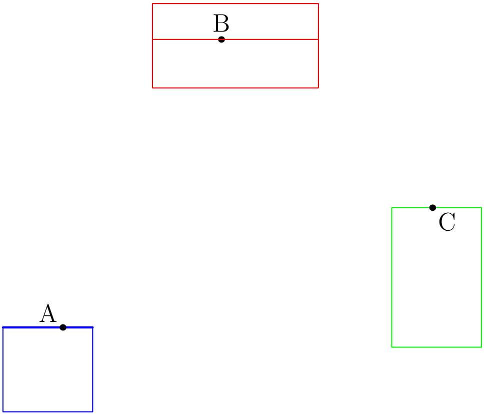
Contracteur triangle



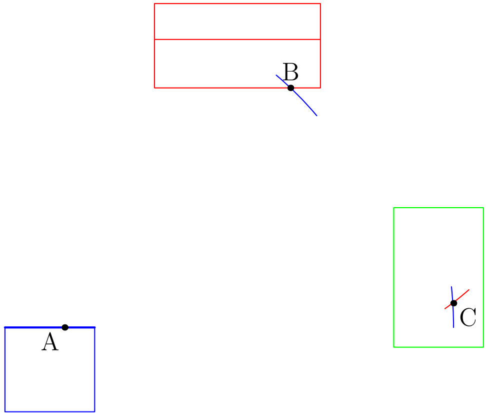
Contracteur triangle



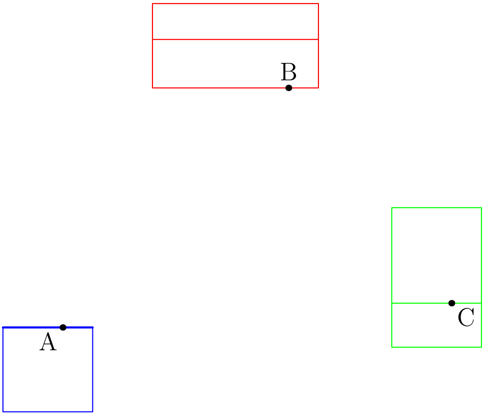
Contracteur triangle



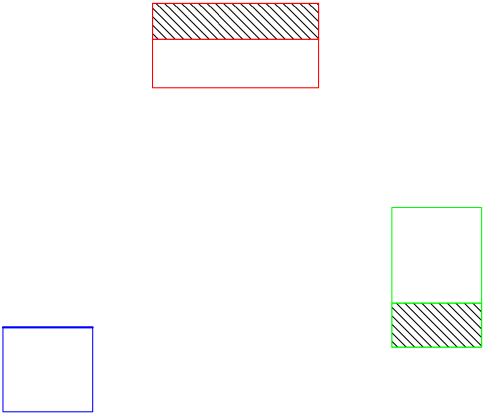
Contracteur triangle



Contracteur triangle



Contracteur triangle



Conclusion

In this presentation, we have proposed a method for contracting boxes [A], [B], and [C] while remaining consistent with the ABC constraint, forming a triangle. We have first contracte the boxes with the distance contarctor. After, using the property that if two points on the boundary of the solution are selected, then the third point is also on the boundary, we propose a formal method to improve the border of the solution.

Perspective

In the following, we will attempt to generalize this method to convex polygons with more than 3 sides. We will then explore combining this approach with the Lasserre method.

THANK FOR YOUR ATTENTION.