Stabilization in finite time for a thermal system described by a parabolic partial differential equation

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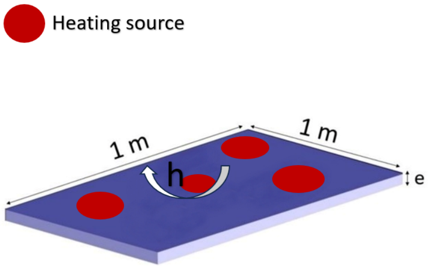
The physical problem

- $1m^2$ steel plate, of $2mm$ thickness (e)
- 4 fixed heating disks (3 big, 1 small)
- Natural convection (h)
- 8 pointwise sensors (infrared cameras)



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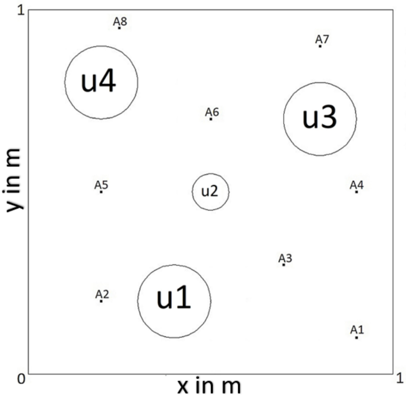

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- $\psi(x, y, t) = \theta(x, y, t) - \theta_{target}(x, y)$: error function (°C)

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- Γ : boundary of Ω

Problem statement

Plate temperature satisfies the conduction-convection PDE:

$$\rho C \frac{\partial \theta}{\partial t} - \lambda \Delta \theta = \frac{-h(\theta - \theta_0) + u(x,y,t)}{e} \qquad \text{on } \Omega \times T$$

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where

$$u(x,y,t) = \sum_{i=1}^{4} u_i(t) \mathbf{1}_{\mathcal{D}_i}(x,y)$$

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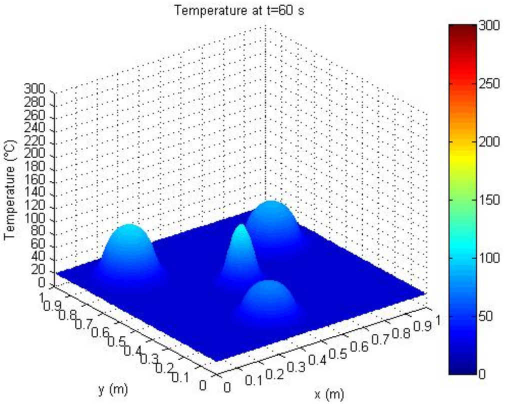
- $t_f = 3600 \ s$

Temperature distribution for several times



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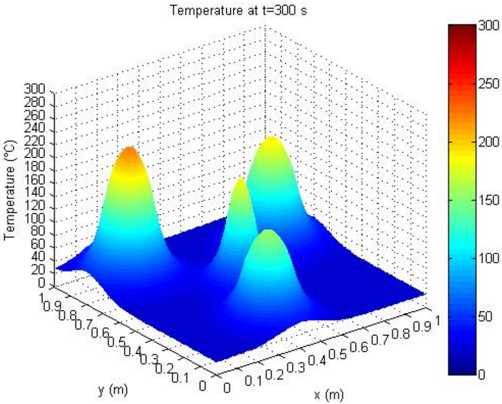
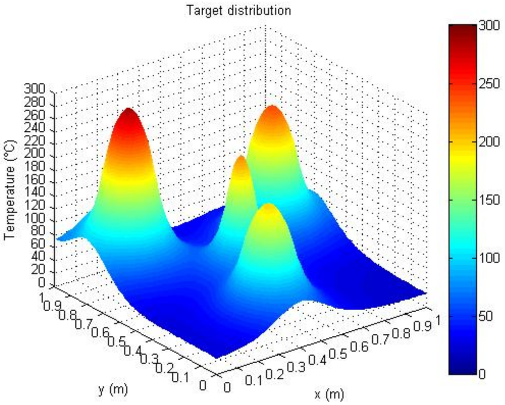
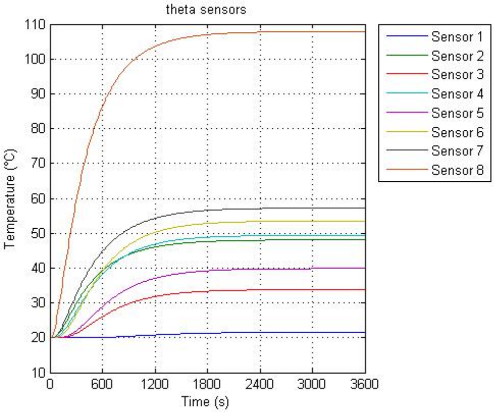


Target distribution at $t = 3600\ s$



Eight pointwise sensors measurements

The pointwise sensors permit to obtain temperature variation over $T = [0, 3600]$



Objectives

- Define $\psi(x, y, t) = \theta(x, y, t) - \theta_{target}(x, y)$

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- CGM objective : identify the control flux $u(x, y, t)$ necessary for minimizing :

$$J(\psi, u) = \frac{1}{2} \sum_{i=1}^{8} \psi(A_i, t_f^*; u)^2$$

Objectives

where:

- $u(x, y, t) = \sum_{i=1}^{4} \sum_{j=1}^{N_t} u_{ij} s_j(t) \mathbf{1}_{\mathcal{D}_i}(x, y)$

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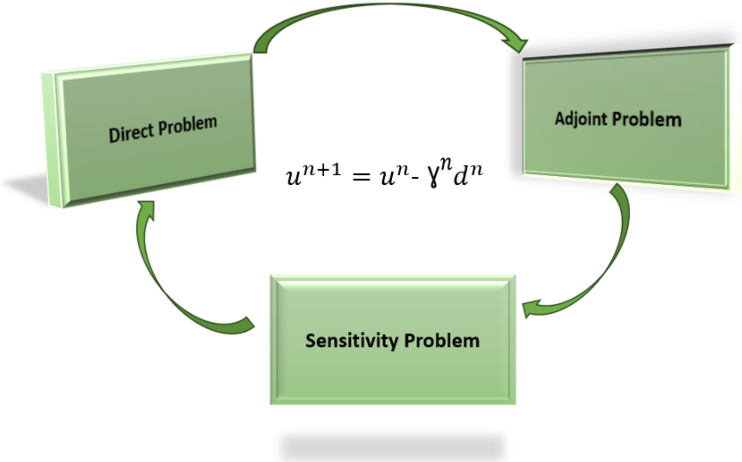
The control $u(x, y, t)$ is determined by the knowledge of $[u_{ij}]_{\substack{i=1,\dots,4 \\ j=1,\dots,N_t}}$

Schema for CGM steps



Step 1: Initialization

Conjugate Gradient Method steps are outlined as follows :

- Initialization of $[u_{ij}]^0_{\substack{i=1,\dots,4 \\ j=1,\dots,N_t}}$ at iteration $n=0$

Step 2: The direct problem

- Simulation on $\Omega \times T^*$:

$$\rho C \frac{\partial \psi^n}{\partial t} - \lambda \Delta \psi^n = \lambda \Delta \theta_{target} + \frac{-h(\psi^n - \psi_0) + u^n(x,y,t)}{e}$$

$$\psi(x,y,0) = \psi_0(x,y) = \theta_0(x,y) - \theta_{target}(x,y)$$

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➢ If not, continue to the next step

Step 3: The adjoint problem

- Simulation of the Lagrange multiplier on $\Omega \times T^*$

$$\rho C \frac{\partial \varphi^n}{\partial t} + \lambda \Delta \varphi^n = \frac{h\varphi^n}{e}$$

$$\varphi^n(x, y, t_f^*) = -(\rho C)^{-1} \sum_{i=1}^{8} \psi^n(x, y, t_f^*) \delta_{A_i}(x, y)$$

$$-\lambda \frac{\partial \varphi^n}{\partial \vec{n}} = 0 \quad \text{on } \Gamma \times T^*$$

Step 3: The adjoint problem

- Calculation of

$$\frac{\partial J}{\partial u_{ij}^n} = -\int_{T^*}\int_{\mathcal{D}_i} \frac{\varphi^n(x,y,t)}{e} s_j(t)dxdydt$$

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$$\frac{\partial J}{\partial u_{ij}^n} = -\int_{T^*}\int_{\mathcal{D}_i} \frac{\varphi^n(x,y,t)}{e} s_j(t)dxdydt$$

- Deduction of $\nabla J^n = \left[\frac{\partial J}{\partial u_{ij}^n}\right]_{\substack{i=1,\ldots,4 \\ j=1,\ldots,N_t}}$ and $\beta_n = \frac{\|\nabla J^n\|_F^2}{\|\nabla J^{n-1}\|_F^2}$

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- Deduction of the descent direction

$$d^n = -\nabla J^n + \beta_n d^{n-1}$$

Step 4: The Sensitivity problem

- Simulation on $\Omega \times T^*$:

$$\rho C \frac{\partial \delta\psi^n}{\partial t} - \lambda \Delta \delta\psi^n = \frac{-h\delta\psi^n + \delta u^n}{e}$$

$$\delta\psi^n(x, y, 0) = 0$$

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- Deduction of the descent depth:

$$\gamma^n = \frac{\sum_{i=1}^{8} \psi(A_i, t_f^*; \mathbf{u}^n)\delta\psi_{\mathbf{d}^n}(A_i, t_f^*; \mathbf{u}^n)}{\sum_{i=1}^{8} \delta\psi_{\mathbf{d}^n}(A_i, t_f^*; \mathbf{u}^n)^2}$$

Step 5: Update the iteration

- Update the control $u^{n+1} = u^n - \gamma^n d^n$, and return to step 2 (the direct problem)

First study: $t_f^* = 600\ s$

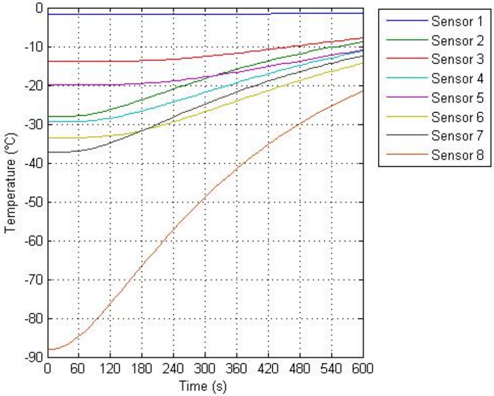

Figure: Error function sensors without control

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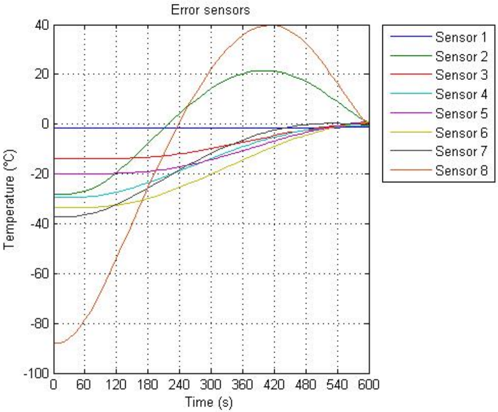

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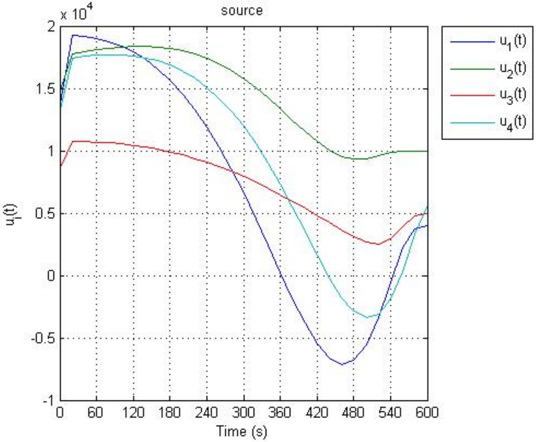

Figure: Variation of the 4 controllers

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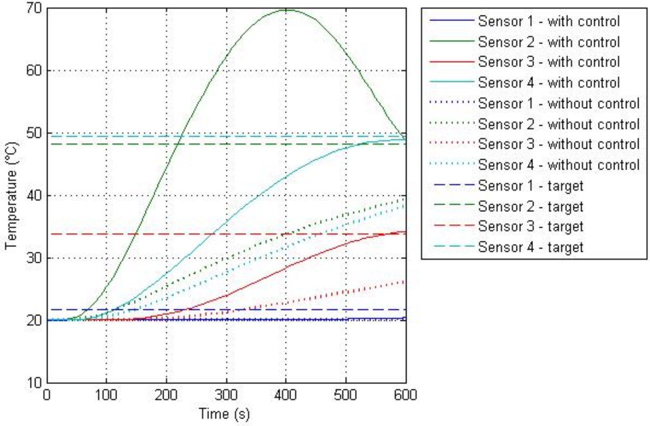
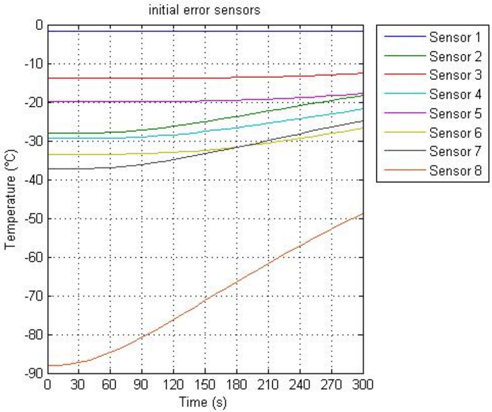


Figure: Temperature sensors with and without control, compared to target

Second study: $t_f^* = 300\ s$



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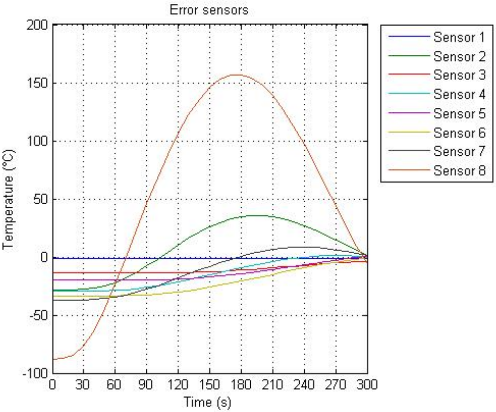

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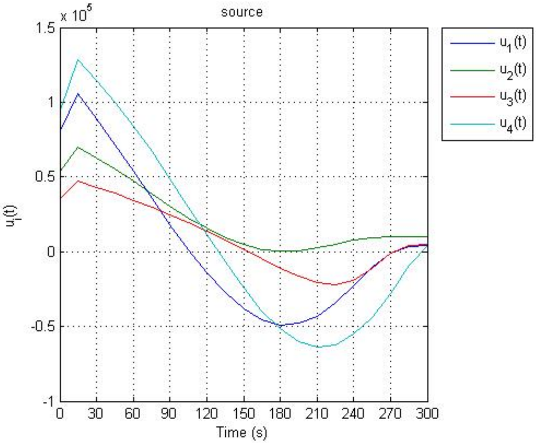

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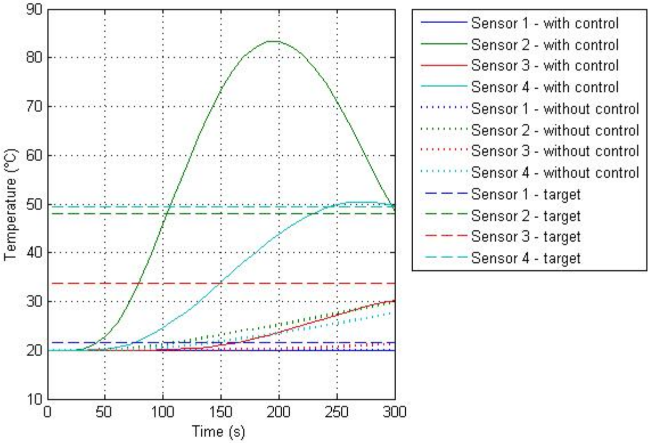

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Conclusions and perspectives

- Implement MGC for $2D$ parabolic PDEs in the presence of external disturbances (offline then quasi-online)
- Design a real $2D$ geometry experiment with infrared sensors and fixed heating disks.
- Explore theoretical developments such as general mixed-type Robin boundary conditions, non-linear parameters (heat capacity, diffusivity, reaction coefficients and boundary heating flux), mobile sensors and actuators, adaptation of control in cases of failure of actuators or sensors and the extension to other types of partial differential equations such as hyperbolic PDEs.

Conclusions and perspectives

- Develop advanced control strategies and algorithms for future research that can improve system performance and stability in the field of $2D$ thermal modeling.
- Submit theoretical developments in "Automatica" and "IEEE Transactions on Automatic Control".
- Present the experimental validation of the methodology on a real thermal process in "Control engineering practice" and "Journal of process control".
- Present studies at well-known conferences: ECC, CDC, CPDE.

Thanks for your attention!

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